Week 9 :, Chapter-5
Eigen values and Eigen vectors

Defn: Let A be a square matrix of order n, then the value of λ for which the equation $Ax = \lambda x$ — ① has a non-trivial solution are called Eigen value of A.

corresponding Eigen value λ there exist a non-zero vector X. such that $|A - \lambda I| X = 0$ Then X is called The Eigen vector.

Remark ① $|A - \lambda I| = 0$ is called the characteristic equation (in some book) $|\lambda I - A| = 0$ or $\det(\lambda I - A|$

② The Eigen value of triangular matrix are its diagonal elements.

③ A is singular matrix $|A| = 0 \Leftrightarrow \lambda = 0$

Q1. Determine the Eigen value and the corresponding Eigen vector of $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$

Soln: **Step 1:** To find the Eigen value first write the characteristic equation

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -6 & \lambda - 2 \end{vmatrix} = 0 \quad — ①$$

$$(\lambda - 3)(\lambda - 2) - 6 = 0$$
$$\lambda^2 - 2\lambda - 3\lambda + 6 - 6 = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\lambda(\lambda - 5) = 0$$
$$\lambda = 0, 5$$

Thus $\lambda_1 = 0$ and $\lambda_2 = 5$ are the Eigen values of A.

Step 2: To find the Eigen Vector

(i) Eigen vector corresponding to $\lambda_1 = 0$

$(\lambda_1 I - A) X = 0$ Put $\lambda_1 = 0$ in ①

$$\Rightarrow \begin{pmatrix} 0-3 & -1 \\ -6 & 0-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -3 & -1 \\ -6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad R_2 \to R_2 + 2R_1$$

$$\begin{pmatrix} -3 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Since the The rank of the matrix is one. So there is only one Leading variable ie x_1 and one free variable ie x_2. So we can write

$$-3x_1 - 1x_2 = 0 \quad \text{or}$$

$$-3x_1 = x_2$$

Let us assume $x_2 = 3$ Then $x_1 = -\frac{\cancel{3}}{\cancel{3}} = -1$

So Eigen vector is $(-1, 3)$

Let us assume $x_2 = 6$ Then $x_1 = -\frac{\cancel{6}\,2}{\cancel{3}}$

So another Eigen vector may be $(-2, 6)$ and so on.

(ii) Eigen vector corresponding to $\lambda = 5$

we have $(\lambda I - A) X = 0$ Put in ①

$$\begin{bmatrix} 5-3 & -1 \\ -6 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + 3R_1$
Since Rank is one
$2x_1 - x_2 = 0$
Leading variable x_1
free variable x_2

$2x_1 = x_2$ Let $x_2 = 2$ Then $x_1 = \frac{2}{2} = 1$ $(1, 2)$ is the Eigen vector

Similarly let $x_2 = 4$ Th

Q2. Find the Eigen value of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

Soln: The characteristic equation is

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda - (-1) \end{vmatrix} = 0$$

$$(\lambda - 3) \quad (\lambda + 1) = 0$$

So $\lambda = 3$ and $\lambda = -1$ are the Eigen value of A.

Q3. Find the Eigen value of $A = \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$

Soln: Since it is a Lower triangular matrix. So by the property its diagonal elements are the Eigen value:

$\lambda_1 = 1/2$, $\lambda_2 = 2/3$ $\lambda_3 = 1/4$

Eigen value of 3 × 3 Matrix

Q4. Find the Eigen value of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

Soln: The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 0 & -1 & -0 \\ -0 & \lambda - 0 & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

open the determinant along Row side

$$= \lambda \begin{vmatrix} \lambda & -1 \\ 17 & \lambda - 8 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ -4 & \lambda - 8 \end{vmatrix} + 0 \begin{vmatrix} \end{vmatrix} = 0$$

$$\lambda \left[(\lambda^2 - 8\lambda) + 17 \right] + (0 - 4) + 0 = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \quad -(1)$$

Now check $\lambda = \pm 1$, ± 2, ± 3, ± 4 Successively Substitute These value in ① So we Observe at $\lambda = 4$ it will satisfy ① So $(x-4)$ is one solution Now

$$\begin{array}{r} \lambda^2 - 4\lambda + 1 \\ \lambda - 4 \overline{) \lambda^3 - 8\lambda^2 + 17\lambda -} \\ \underset{-}{\lambda^3} \underset{+}{-} 4\lambda^2 \\ \hline -4\lambda^2 + 17\lambda \\ \underset{+}{-4\lambda^2} \underset{-}{+} 16 \\ \hline \cancel{\lambda} - \\ \underset{-}{\cancel{\lambda}} \underset{+}{-4} \\ \hline 0 \; 0 \end{array}$$

$$(\lambda - 4)\ (\lambda^2 - 4\lambda + 1) = 0$$

Now I am going to factorise with $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \lambda = 2 - \sqrt{3}$$

are the Eigen value.

Q Find the Eigen value, Eigen vector and diagonalize the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Soln: The characteristic equation is $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0 \quad —①$$

on solving $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

check for $\lambda = 1$ $\lambda^3 - 5(1)^2 + 8(1) - 4 = 0$

$0 = 0$

$(\lambda - 1)$ is a factor Now

$$\lambda - 1 \overline{)\ \lambda^3 - 5\lambda^2 + 8\lambda - 4\ }(\lambda^2 - 4\lambda + 4$$

$$\underline{\pm\lambda^3 \mp \lambda^2}$$

$$-4\lambda^2 + 8\lambda$$

$$\underline{-4\lambda^2 + 4\lambda}$$

$$+ \quad -$$

$$4\lambda - 4$$

$$\underline{4\lambda \mp 4}$$

$$0$$

So

$(\lambda - 1)(\lambda^2 - 4\lambda + 4)$

$(\lambda - 1)(\lambda - 2)(\lambda - 2)$

$(\lambda - 1)(\lambda - 2)^2$

$\lambda_1 = 1, \quad \lambda_2 = 2, 2$

are the Eigen value

Eigen vector: corresponding to $\lambda = 1$

$(\lambda I - A)X = 0$ from ①

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \rightarrow R_2 + R_1$
$R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} \textcircled{1} & 0 & 2 \\ 0 & \textcircled{-1} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since the rank of the matrix is 2. There are two Leading variable and one free variable: Leading x_1 and x_2, free x_3

Now we can write

$x_1 + 0x_2 + 2x_3 = 0$ or $x_1 + 2x_3 = 0 \quad —②$

$-x_2 + x_3 = 0 \qquad x_3 = x_2 \quad —③$

Let $x_3 = 1$ in ③ Then $x_2 = 1$ and from ② $x_1 = -2x_3 \Rightarrow x_1 = -2$

Eigen vector corresponding to $\underline{\lambda = 2}$

from ①

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \frac{R_1}{2}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since the rank is 1
The leading variable is x_1
The free variables are x_2, x_3

Now

$$x_1 + 0x_2 + x_3 = 0$$

or

$$x_1 = -x_3$$

(i) ~~Put x~~ Let $x_3 = 1$ Then $x_1 = -1$ Let assume $x_2 = 0$

so Eigen vector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(ii) Let $x_3 = 0$ Then $x_1 = 0$ Let assume $x_2 = 1$

So Eigen vector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now we have Three Eigen vectors corresponding to $\lambda = 1$ and $\lambda = 2$
Let us call them P_1 P_2 and P_3. such that $P_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} - \\ 1 \\ 1 \end{bmatrix}$
Now The Matrix P diagonalizes A. where P

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

We can check and verify $P^{-1}AP =$
The diagonal elements are the Eigen values

$^{-1}$ n $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$ — Eigen value

: Consider $A = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & 6 & 1 \end{bmatrix}$

Sol: The characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda-4 & -6 & 0 \\ 3 & \lambda+5 & 0 \\ 3 & 6 & \lambda-1 \end{vmatrix} = 0 \quad \text{——} ①$$

Open the determinant from the last column

$$0 \begin{vmatrix} \; \end{vmatrix} - 0 \begin{vmatrix} \; \end{vmatrix} + \lambda-1 \begin{vmatrix} \lambda-4 & -6 \\ 3 & \lambda+5 \end{vmatrix} = 0$$

$$(\lambda-1) \left[(\lambda-4 \; (\lambda+5) + 18 \right] = 0$$

$$(\lambda-1) \{ \lambda^2 + 5\lambda - 4\lambda - 20 + 18] = 0$$

$$(\lambda-1) [x^2 + \lambda - 2] = 0$$

$$(\lambda-1) [\lambda^2 + 2\lambda - \lambda - 2] = 0$$

$$(\lambda-1) \{ (\lambda-1)(\lambda+2)] = 0$$

$$(\lambda-1)^2 (\lambda+2) = 0$$

The Eigen values are $\lambda_1 = 1$, $\lambda_2 = -2$

The Eigen vectors corresponding the Eigen value

Now The Eigen vectors

① $\lambda_1 = 1$

$$(\lambda I - A) X = 0 \quad \begin{bmatrix} 1-4 & -6 & 0 \\ 3 & 1+5 & 0 \\ 3 & 6 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & -6 & 0 \\ 3 & 6 & 0 \\ 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} -3 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad -\frac{R_1}{3}$$

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Since the rank of the matrix is 1 so there is only one leading variable and 2 free variable.

The Leading variable is x_1 $\therefore$ Since it has a leading 1.
and the free variables are x_2 and x_3

$x_1 + 2x_2 = 0$ or $x_1 = -2x_2$ — ②

(i) Let $x_2 = 1$ and $x_3 = 2$ (cos these are free choose any) Now from ② $x_1 = -2$, $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$

(ii) Let $x_2 = -2$ and $x_3 = +2$ Then from ② $x_1 = 4$, Then $\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$

(iii) Let $x_2 = 3$ and $x_3 = 4$ Then from ② $x_1 = -6$ Then $\begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix}$

Similarly corresponding $\lambda = -2$ Put in ①

$$(\lambda I - A) = 0$$

$$\begin{bmatrix} -2-4 & -6 & 0 \\ 3 & -2+5 & 0 \\ 3 & 6 & -2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -6 & -6 & 0 \\ 3 & 3 & 0 \\ 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \frac{R_1}{-6}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & 0 \\ 3 & 6 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \quad \frac{R_2}{3}$$

$$\begin{bmatrix} \textcircled{1} & 1 & 0 \\ 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since The rank of the matrix is 2 so there are 2 ~~free~~ ~~variables and~~ Leading variables an 1 free variable

Leading varible x_1 and x_2 ∴ coz it hase leading 1

free varible x_3

$$x_1 + x_2 = 0 \qquad \text{or} \qquad x_1 = -x_2$$

$$x_2 - x_3 = 0 \qquad\qquad x_2 = x_3$$

So choose any value of x_3

Let $x_3 = 1$ Then $x_2 = 1$ and $x_1 = -1$

Let $x_3 = 2$ Then $x_2 = 2$ and $x_1 = -2$

$x_3 = 3$ $x_2 = 3$ $x_1 = -3$

Diagonalization

Let $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ Then

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 5^2 & 0 \\ 0 & 3^2 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^3 = \begin{pmatrix} 5^3 & 0 \\ 0 & 3^3 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ 0 & 27 \end{pmatrix}$$

$$D^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 5^{-1} & 0 \\ 0 & 3^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1/3 \end{pmatrix}$$

Which matrix is diagonalizable

① An nxn matrix is diagonolisable iff it has n L.I, E.V.

② If A has distinct Eigen values.

③ If A is Symmetric.

④ The matrix A and its diagonalization have the Same Eigen values.

Q2. Diagonalise the following matrix if Possible

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 3 \end{pmatrix}$$

The E. Values are $\lambda = 1, 1, 3$

it is ~~do~~ not diagonalizable. Since E. value are not distinct.
and the matrix is not Symmetric

Q3. Diagonalize the matrix if Possible $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$

Sol: The Eigen values are ~~x~~. (∴ Since the matrix is Symmetric Proce

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -3 \\ -3 & \lambda - 6 \end{vmatrix} = 0 \qquad \lambda_1 = -7,\ \lambda_2 = +3$$

are distinct

$$(\lambda I - A)x = 0 \qquad = \begin{pmatrix} -7-2 & -3 \\ -3 & -7-6 \end{pmatrix} = \begin{pmatrix} -9 & -3 \\ -3 & -13 \end{pmatrix} \frac{R_1}{9} \times R_2$$

$$= \begin{pmatrix} 1 & 1/3 \end{pmatrix} R_2 \to R_2 - 3R_1 = \begin{pmatrix} 1 & 1/3 \\ 0 & 12 \end{pmatrix} \frac{R_2}{12}$$

$=$

$$= \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Since the ~~ore~~ → the rank is 2 and 2 leading variable There is no free varible so

$$x_1 + \tfrac{1}{3} x_2 = 0 \qquad \Rightarrow \quad x_1 = 0$$

$$x_2 = 0$$

Similarly $\lambda = 3$

Since $\lambda_1 = -7$ and $\lambda_2 = -3$

so $D = \begin{pmatrix} -7 & 0 \\ 0 & 3 \end{pmatrix}$ Since The matrix A and its diagonaliz
have the same E. Values. (Theore

E. vectors are

$$P = \begin{pmatrix} -1/3 & 3 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \frac{1}{10} \begin{pmatrix} -3 & 9 \\ 3 & 1 \end{pmatrix}$$

Now ~~$A = P^{-1} A P$~~ $D = P^{-1} A P$

$$= \frac{1}{10} \begin{pmatrix} -3 & 9 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} -1/3 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 0 \\ 0 & 3 \end{pmatrix}$$