

Matrix Properties

- (a) $A + B = B + A$ commutative law for addition
- (b) $A + (B + C) = (A + B) + C$ (Associative law for addition)
- (c) $A (BC) = (AB) C$ Associative Law for multiplication

Note: In Matrix $AB \neq BA$

for ex: Consider $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Then $AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$ and $BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$

Thus $AB \neq BA$

Zero Matrix is denoted by O for ex:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1 \times 3}$ etc $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$A + O = A$

$A + (-A) = O$

$O(A) = O$

Identity matrix an $n \times n$ matrix with ones on the main diagonal and zeros elsewhere for ex:

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Invertible (non singular): If $AB = BA = I$, then A is invertible for ex:

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus A and B are invertible and each is an inverse of the other.

Theorem 1.4.5 Inverse (old method)

If the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if

$ad - bc \neq 0$ Then the formula is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For ex: (a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

(a) The determinant $\det(A) = (6)(2) - (1)(5) = 7$

which is non-zero, Thus A is non-singular or invertible or we can solve it

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

(b) $\det(A) = (-1)(-6) - (2)(3) = 0$

we can't solve it.

Using the row operation find the inverse (New method) ⁽²⁾₂

Q. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$

Sol: Now we can write

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array} \right]$$

So the final matrix will be of the form

$$[I | A^{-1}] \quad \text{Now lets start}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array} \right] R_2 \rightarrow R_2 + 2R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array} \right] R_2 \rightarrow -R_2$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array} \right] R_1 \rightarrow R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & -2 & -1 \end{array} \right]$$

So we have

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$$

You can check by multiply each other we get the identity matrix.

$$A A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex 4: Using Row operation e- find A^{-1} , $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Soln:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

~~$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & +1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_1 \times -ve \end{array}$$~~

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 + 2R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right] \times -R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

for each $n \times 1$ matrix b , the system of equations $Ax = b$ has exactly one solution, namely $x = A^{-1}b$.

Q. Find the solution of system of linear equations using A^{-1} .

$$\begin{aligned}x + 3y &= 1 \\ 2x + 5y &= 3\end{aligned}$$

Soln:

we can write it like

$$A X = b$$

\downarrow coefficient matrix \downarrow variable matrix \downarrow constant matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

So $Ax = b$

or $A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

or $x = A^{-1}b$ — (1)

Now

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

$$[A | I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array} \right] \quad R_2 \rightarrow R_2 - 2R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array} \right] \quad R_2 \times -ve$$

$$= \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array} \right] \quad R_1 \rightarrow R_1 - 3R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array} \right]$$

$$\text{So } A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

Now from (1)

$$A^{-1}b = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

So from (1)

$$x = 4 \quad \text{and} \quad y = -1$$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called lower triangular matrix.

and a square matrix in which all the entries below the main diagonal are zero are called upper triangular matrix.

For ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

upper triangular matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Lower triangular matrix

Q2 Apply the elementary row operations to find the inverse of (4)

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$. And use the result to solve the system

of equations

$$\begin{aligned} x + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 7x_3 &= 0 \\ 3x_1 + 7x_2 + 9x_3 &= 1 \end{aligned}$$

Step 1: find A^{-1}

$$[A/I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_1 \rightarrow R_1 - 2R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right] \begin{array}{l} R_3 (-1) \\ R_1 \rightarrow R_1 + R_3 \\ R_2 \rightarrow R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right]$$

Step 2: Solve the system $Ax = B$ or $x = A^{-1}b$:

$$x = \begin{bmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

inconsistent (no solution), Unique solution, infinitely many solutions

$$x_1 + ax_2 = 1, \quad x_1 + 2x_2 = b$$

Soln: Write the Augmented matrix

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 1 & 2 & b \end{array} \right] \quad R_2 \rightarrow R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 2-a & b-1 \end{array} \right]$$

There are two cases:

Case 1: $a = 2$ Then

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & b-1 \end{array} \right]$$

- (i) If $b \neq 1$ Then No solution (inconsistent)
- (ii) If $b = 1$ Then Infinitely many solutions

Case 2: $a \neq 2$ Then

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 2-a & b-1 \end{array} \right] \quad R_2 \rightarrow \frac{R_2}{2-a}$$

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 1 & \frac{b-1}{2-a} \end{array} \right]$$

It has unique solution for all b .