Week-12, chapter 9
Numerical Methods

LU-Decomposition:

step 1: Reduce the matrix to row Echelon form

step 2: In each position along the main diagonal of L, Place the reciprocal of the multiplier

step 3: In each Position ,, ,, Place the negative of the multiplier

step 4: Form the decomposition $A = LU$

Q. Find an LU-decomposition

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

Sol: Given

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad R_1 \to \frac{R_1}{6}$$

x denote the unknown entry of I

$$\begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \quad \begin{array}{l} \text{multiplier} = \frac{1}{6} \\ R_2 \to R_2 - 9R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \quad \begin{array}{l} \text{multiplier} = -9 \\ \text{multiplier} = -3 \end{array}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & x & 0 \\ 3 & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 5 \end{bmatrix} \quad \text{multiplier} = \frac{1}{2} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{multiplier} = -8 \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & x \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{multiplier} = 1 \qquad L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$$

So we have constructed the LU-decomposition

$$A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

→ See Q2 Next page.

Q2. ~~Power~~ The Power Method

step 1: choose an arbitrary non-zero vector

step 2: compute Ax_0 and multiply it by $\frac{1}{\max(Ax_0)}$

step 3: compute Ax_1 and scale it by factor $\frac{1}{\max(Ax_1)}$

Similarly and so on

Q. Apply the Power Method with maximum entry scaling to $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ stop at x_5

Soln:

$$Ax_0 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and } x_1 = \frac{Ax_0}{\max(Ax_0)}$$

So $x_1 = \frac{1}{3}\begin{bmatrix}3\\2\end{bmatrix} \approx \begin{bmatrix}1.0000\\0.66667\end{bmatrix}$

Now

$$Ax_1 = \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}1.00000\\0.66667\end{bmatrix} \approx \begin{bmatrix}4.33333\\4.00000\end{bmatrix} \text{ and}$$

$$x_2 = \frac{Ax_1}{max(Ax_1)} \approx \frac{1}{4.33333}\begin{bmatrix}4.33333\\4.00000\end{bmatrix} \approx \begin{bmatrix}1.00000\\0.92308\end{bmatrix}$$

$$Ax_2 = \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}1.00000\\0.92308\end{bmatrix} \approx \begin{bmatrix}4.84615\\4.76923\end{bmatrix} \text{ and}$$

$$x_3 = \frac{Ax_2}{max(x_2)} = \frac{1}{4.84615}\begin{bmatrix}4.84615\\4.76923\end{bmatrix} \approx \begin{bmatrix}1.00000\\0.98413\end{bmatrix}$$

$$Ax_3 = \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}1.00000\\0.98413\end{bmatrix} \approx \begin{bmatrix}4.96825\\4.95238\end{bmatrix} \text{ and}$$

$$x_4 = \frac{Ax_3}{max(Ax_3)} = \frac{1}{4.96825}\begin{bmatrix}4.96825\\4.95238\end{bmatrix} = \begin{bmatrix}1.000\\0.996\end{bmatrix}$$

$$Ax_4 = \begin{bmatrix}3 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}1.00000\\0.99681\end{bmatrix} \approx \begin{bmatrix}4.99361\\4.99042\end{bmatrix}$$

$$x_5 = \frac{Ax_4}{max(Ax_4)} = \frac{1}{4.99361}\begin{bmatrix}4.99361\\4.99042\end{bmatrix} \approx \begin{bmatrix}1.000\\0.9936\end{bmatrix}$$

Now

$$\lambda_1 = \frac{(Ax_1)^T x_1}{x_1^T x_1} \approx \frac{7.00000}{1.44444} \approx 4.84615$$

$$\lambda_2 = \frac{(Ax_2)^T x_2}{x_2^T x_2} \approx \frac{9.24852}{1.85207} = 4.99361$$

$$\lambda_3 = \frac{(Ax_3)^T x_3}{x_3^T x_3} \approx \frac{9.84203}{1.96851} = 4.99974$$

$$\lambda_4 \quad \frac{(Ax_4)^T x_4}{x_4^T x_4} \approx \frac{9.96808}{1.99362} \approx 4.99999$$

$$\lambda_5 = \frac{(Ax_5)^T x_5}{x_5^T x_5} \cong \frac{9.99360}{1.99872} \cong 5.00000.$$

Singular value Decomposition: If A is $m \times n$ matrix and if $\lambda_1, \lambda_2 \ldots$ are the E.V Eigen value of $A^T A$. Then the numbers $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}, \ldots$ are called The Singular Values of A.

Q1 Find ~~a~~ Singular value decomposition of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Soln: First find $A^T A$ ie

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

~~The characteristic~~ Now to find the Eigen value of $A^T A$.

The characteric Eqn is $|\lambda I - A^T A| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$4 - 2\lambda - 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3, 1$$

So $\lambda_1 = 3$ and $\lambda_2 = 1$ are the Eigen value so

The singular value of A is

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$$

Now find the E.V corresponding to $\lambda_1 = 3$ & $\lambda_2 = 1$ are

$$v_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Now to find u_i where $u_i = \frac{1}{\sigma_i} A v_i$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \sqrt{6}/6 \\ \sqrt{6}/6 \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

with the help of u_1 and u_2 we can find u_3

$$u_3 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Now the singular value decomposition of A is

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^T$$

$v_1 \quad v_2$

Singular values are in diagonal

Q2 Find an LU-decombosition of $A = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$

Sol:

$$A = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} \quad \frac{R_2}{2} \rightarrow \quad \begin{bmatrix} x & 0 \\ x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1 \quad \begin{bmatrix} 2 & 0 \\ x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \quad \frac{R_2}{3} \quad \begin{bmatrix} 2 & 0 \\ -1 & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

So

$$\begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

Q find an LU decomposition of the following matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$

Soln:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} x & 0 \\ x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} R_2 \to R_2 - 2R_1 \begin{bmatrix} 1 & 0 \\ x & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} R_2(-1) \quad \begin{bmatrix} 1 & 0 \\ 2 & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

So

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Q find the singular value of the matrix A if given the foll

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Soln: The characteristic equation is $|\lambda I - A^T A| = 0$

$$\begin{vmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{vmatrix} = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$\lambda = 1, 2, 3$ are the Eigen values

Then the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{1} = 1$
$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$, $\sigma_3 = \sqrt{\lambda_3} = \sqrt{3}$

Q Find the distinct singular values of $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Soln: First we find

$$A^T A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

The characteristic equation is

$$|\lambda I - A^T A| = \begin{bmatrix} \lambda-5 & 0 \\ 0 & \lambda-5 \end{bmatrix}$$

$$\Rightarrow (\lambda-5)(\lambda-5) = 0$$

$\lambda = 5, 5$ are the E.V.

So the singular value of A is

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{5}$$

Q Use the method of LU decomposition to solve the system

$$2x - 4y = -2$$
$$-x + 3y = 2$$

Soln: we can write

$$\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

LU decomposition of A.

$$\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} \frac{R_1}{2} \quad \begin{bmatrix} x & 0 \\ x & x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} R_1 + R_2 \begin{bmatrix} 2 & 0 \\ x & x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

So

$$\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

Using $LY = B$

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

we set $y_1 = -1$, $y_2 = 1$

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

we set $x = 1$, $y = 1$