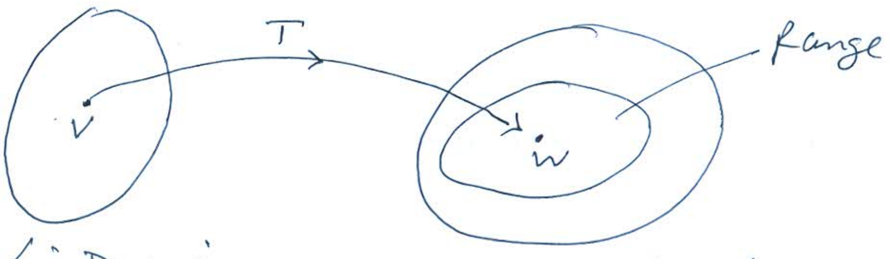
Week-11, chapter 8

Linear Transformations



V: Domain W: codomain

$$T: V \to W$$

Function T that maps a vector space V into a vector space W:

Q1 $T: R^2 \to R^2$ $v = (v_1, v_2) \in R^2$, $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$ — (1)

(a) Find the image of $v = (-1, 2)$ (b) Find the Pre-image of $w = (-1, 11)$

Soln: (a) $v = (-1, 2)$

$$T(v) = T(-1, 2) = (-1-2, -1+2(2)) \quad \text{from (1)}$$
$$= (-3, 3)$$

(b) $T(v) = w = (-1, 11)$ $\quad v \xrightarrow{T} w$

we know $T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$

So $v_1 - v_2 = -1$

$v_1 + 2v_2 = 11$

So $v_1 - v_2 = -1$

$v_1 = 3$, $v_2 = 4$

-se of $(-1, 11)$

-ransformation: Let V, W be the Vector Space

$V \to W$: V to W linear Transformation if

$T(u+v) = T(u) + T(v)$ $\forall u, v \in V$

Linear T

T

Q1 Verify a Linear Transformation T from R^2 into R^2

$$T(v_1, v_2) = (v_1 - v_2,\ v_1 + 2v_2) \quad — (1)$$

Sol: Let $u = (u_1, u_2)$ $v = (v_1, v_2)$ and c is any real No

1 - Let $u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1,\ u_2 + v_2)$

So

~~$T($~~

$$T(u + v) = T(u_1 + v_1,\ u_2 + v_2)$$

$$= ((u_1 + v_1) - (u_2 + v_2),\ (u_1 + v_1) + 2(u_2 + v_2))$$

from (1)

$$= ((u_1 - u_2) + (v_1 - v_2),\ (u_1 + 2u_2) + (v_1 + 2v_2))$$

$$= (u_1 - u_2,\ u_1 + 2u_2) + (v_1 - v_2,\ v_1 + 2v_2)$$

$$= T(u) + T(v) \quad \text{from (1)}$$

2 - Scalar Multiplication

$$cu = c(u_1, u_2) = (cu_1, cu_2)$$

Now

$$T(cu) = T(cu_1, cu_2)$$

$$= (cu_1 - cu_2,\ cu_1 + 2cu_2) \quad \text{from (1)}$$

$$= c(u_1 - u_2,\ u_1 + 2u_2)$$

$$= cT(u)$$

Therefore, T is a Linear Transformation.

Zero Transformation

$T: V \to W \qquad T(v) = 0 \qquad \forall v \in W$

Identity Transformation

$T: V \to V \qquad T(v) = v \qquad \forall v \in V$

Q1 Let $T: R^3 \to R^3$ be a L.T such that

$T(1,0,0) = (2,-1,4)$, $T(0,1,0) = (1,5,-2)$

$T(0,0,1) = (0,3,1)$ Find $T(2,3,-2)$

Sol: we can write

$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$

coz. These are standard basis

Now Apply The Transformation both Side

$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1)$

$= 2(2,-1,4) + 3(1,5,-2) - 2(0,3,1)$

$= (7,7,0)$

Q2 The Function $T: R^2 \to R^3$ is defined by $T(v) = Av =$

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(a) Find $T(v)$ where $v = (2,-1)$

Sol: $v = (2,-1)$

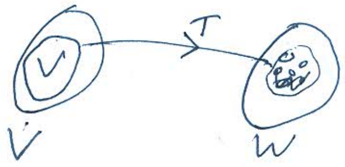
$$T(v) = Av = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underset{R^2 \text{ vector}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \underset{R^3 \text{ vector}}{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$$

$\therefore T(2,-1) = (6,3,0)$

Kernel of a L.T: Let $T: V \to W$ be a L.T Then the set of all vectors v in V that satisfy $T(v) = 0$ is called the Kernel of T and it is denoted by ~~ket~~ ker(T)

$$\ker(T) = \{ v \mid T(v) = 0 \quad \forall v \in V \}$$

Note: Sometimes Kernel T is called the Nullspace of T

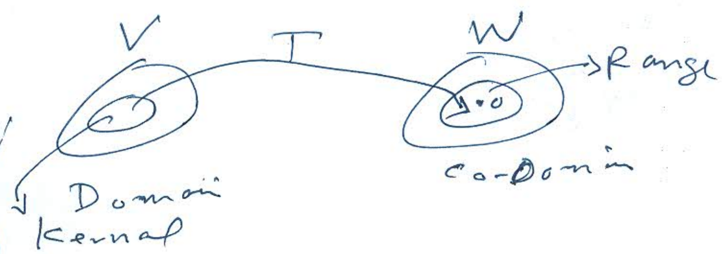


Range of a L.T: Let $T: V \to W$ be a L.T

Then the set of all vectors ~~in V~~ w in W that ~~a~~ are images of vectors in V is called the range of T and denoted by range (T)

Note: (1) Ker(T) is a subspace of V

(2) range (T) is a subspace of W



Rank of a L.T: Let $T: V \to W$

rank (T) = The dimension of the range of T

Nullity of L.T: Let $T: V \to W$

nullity (T) = The dimension of the Kernel of T

Dimension Theorem for a L.T: Let $T: V \to W$ be a L.T

rank (T) + nullity (T) = n

OR

dim (range of T) + dim (Kert of T) = dim (domain of T)

Q1 Find the rank and Nullity of L-T

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Soln: we know rank (T) = rank $(A) = 2$ no. of Non-zero Rows

and nullity (T) = dim (ker T)
= dim (domain of T)
= 3

and given $n = 3$ ∴ matrix is 3×3

So by rank nullity Theorem

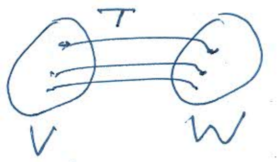
rank (T) + nullity of $T = n$

or nullity of $(T) = n -$ rank (T)

$= 3 - 2$

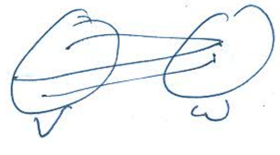
nullity $(T) = 1$

One-to-one: if $T(u) = T(v)$ one-to-one



If pre-image of every w in the range consist of a single vector

Onto: if every element of W has a preimage in V



Theorem: Let $T: V \to W$ be L-T
Then T is 1-1 iff $\ker(T) = \{0\}$

Isomorphism: A L-T $T: V \to W$ That is 1-1 and onto is called isomorphism.

Matrix for L-T

(1) $T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$

OR

(2) $T(x) = Ax = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Q. Finding the standard matrix for the L.T $T : R^3 \to R^2$ defined by $T(x, y, z) = (x - 2y, 2x + y)$ — (1)

Soln:

Vector Notation

$T(e_1) = T(1, 0, 0)$

$= (1 - 2\times 0, 2\times 1 + 0)$ from (1)

$= (1, 2)$

Matrix Notation

$T(e_1) = T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

$T(e_2) = T(0, 1, 0)$

$= (-2, 1)$

$T(e_2) = T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$

$T(e_3) = T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

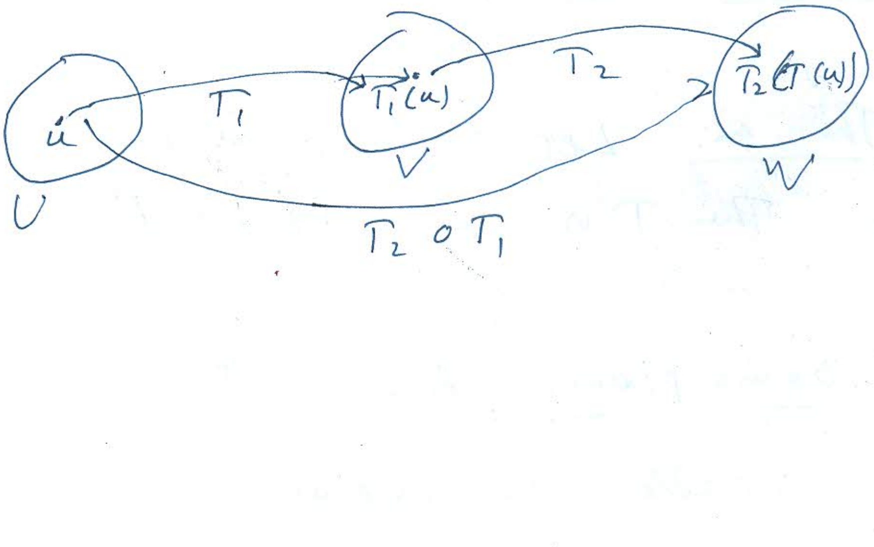
$T(e_3) = T(0, 0, 1) = (0, 0)$

So

$A = [T(e_1), T(e_2)\ T(e_3)]$

$= \begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}$.

Composition



Q. Show that the function $T(A) = A^T$ is linear Transformation

Soln: we know that to show that T is linear we have to show

(i) T Preserve addition:

$$T(A+B) = (A+B)^T$$
$$= A^T + B^T$$
$$= T(A) + T(B)$$

(2) T Preserve scaler multiplication:

$$T(KA) = (KA)^T$$
$$= K A^T$$
$$= KT(A)$$

Q2. Show that the map $T: R^2 \to R^2$ given by $T(x, y) = (x, 0)$ — (1) is linear map

Soln. we know to show T is linear if it preserve addition & scaler multiplication:

(1) T Preserve addition: Take any $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in R^2$

$$T(v_1 + v_2) = T((x_1, y_1) + (x_2, y_2))$$
$$= T(x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2, 0) \quad \text{by (1)}$$
$$= (x_1, 0) + (x_2, 0)$$
$$= T(x_1, y_1) + T(x_2, y_2) \quad \text{by (1)}$$
$$= T(v_1) + T(v_2)$$

(2) T Preserverse scaler multiplication:

$$T(cv) = T(c(x, y))$$
$$= T(cx, cy)$$
$$= (cx, 0) \quad \text{by (1)}$$
$$= c(x, 0) \quad c \text{ is scalar}$$
$$= cT(x, y) \quad \text{by (1)}$$
$$= cT(v)$$

Hence T is a linear map.

Q3 Let $T(x, y, z) = (3x - 2y + z,\ 2x - 3y;\ y - 4z)$
Write down the standard matrix of T, and compute $T(2,-1,-1)$.

<u>Soln:</u>

The matrix will be

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix}$$

And

$$T(2,-1,-1) = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 3 \end{bmatrix}$$

Q Determine whether the function $T: R^2 \to R^2$, where $T(x,y) = (x^2, y)$ — (1) is linear

Soln: we know to show T is linear, we have to show addition & scaler multiplication

(i) T preserve addition: Take $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in R^2$

$$\begin{aligned} T(v_1 + v_2) &= T((x_1, y_1) + (x_2, y_2)) \\ &= T(x_1 + x_2,\ y_1 + y_2) \\ &= ((x_1 + x_2)^2,\ y_1 + y_2) \quad \text{from (1)} \\ &\neq (x_1^2, y_1) + (x_2^2, y) \\ &= T(x_1, y_1) + T(x_2, y_2) \\ &= T(v_1) + T(v_2) \end{aligned}$$

So, T does not preserve additivity, so T is not linear.