Week-9, chapter 6

Inner Product Spaces

Defn: Let u, v, w be vectors in a vector space V, and let c be any scalar; An inner product on V is a function that associate a real number $\langle u, v \rangle$ with each pair of vectors u & v and satisfies the following axioms.

1- $\langle u, v \rangle = \langle v, u \rangle$

2- $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$

3- $c \langle u, v \rangle = \langle cu, v \rangle$

4- $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

Note: A vector space V i.e $(V, +, \cdot)$ with an inner product is called an inner product space $(V, +, \cdot, \langle , \rangle)$

Q: Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in R^2, verify that the Euclidean inner product $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$ satisfies the four inner product axioms

Sol: Axiom 1: $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$
$= 3v_1u_1 + 2v_2u_2$
$= \langle v, u \rangle$

Axiom 2: If $w = (w_1, w_2)$ Then

$\langle u + v, w \rangle = 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2$
$= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2)$
$= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2)$
$= (u, w) + (v, w)$

Axiom 3: $\langle cu, v \rangle = 3(cu_1)v_1 + 2(\cancel{k}u_2)v_2$

Axiom 3: $\langle cu, v \rangle = 3(cu_1)v_1 + 2($

Axiom 4: $\langle v, v\rangle = 3(v_1 v_1) + 2(v_2 v_2)$

$= 3v_1^2 + 2v_2^2 \geq 0$

with equality iff $v_1 = v_2 = 0$

Q2. Show that the function define an inner product in R^2 where $u = (u_1, u_2)$ and $v = (v_1, v_2)$

$\langle u, v\rangle = u_1 v_1 + 2u_2 v_2$

Sol: axiom 1: $u_1 v_1 + 2u_2 v_2 =$

$= v_1 u_1 + 2v_2 u_2$

$= \langle v, u\rangle$

axiom 2: $w = (w_1, w_2)$

$\langle u, v+w\rangle = u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$

$= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2$

$= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2)$

$= \langle u, v\rangle + \langle u, w\rangle$

axiom 3: $c\langle u, v\rangle = c(u_1 v_1 + 2u_2 v_2)$

$= (cu_1)v_1 + 2(cu_2)v_2$

$= \langle cu, v\rangle$

axiom 4: $\langle v, v\rangle = v_1^2 + 2v_2^2 \geq 0$

$\langle v, v\rangle = 0 \Rightarrow v_1^2 + 2v^2 = 0 \Rightarrow v_1 = v_2 = 0$

Properties of inner product

1- $\langle 0, v\rangle = \langle v, 0\rangle = 0$

2- $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

3- $\langle u, cv\rangle = c\langle u, v\rangle$

4- Norm (length of u), $\|u\| = \sqrt{\langle u, u\rangle}$ $\therefore \|u\|^2 = \langle u$

stance b/w u and v

$d(u,v) = \|u-v\| = \sqrt{\langle u-v,\, u-v\rangle}$

Angle b/w two non-zero vectors u & v

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|}, \quad 0 \le \theta \le \pi$$

Note if $u \perp v$ Then $\langle u, v\rangle = 0$ orthogonal

if $\|v\| = 1$ Then v is called the unit vectors

Q calculating the inner products $\langle u-2v, 3u+4v\rangle$

Sol: $\langle u-2v, 3u+4v\rangle = \langle u, 3u+4v\rangle - \langle 2v, 3u+4v\rangle$

$= \langle u, 3u\rangle + \langle u, 4v\rangle - \langle 2v, 3u\rangle - \langle 2v, 4v\rangle$

$= 3\langle u,u\rangle + 4\langle u,v\rangle - 6\langle v,u\rangle - 8\langle v,v\rangle$

$= 3\|u\|^2 + 4\langle u,v\rangle - 6\langle u,v\rangle - 8\|v\|^2$

$= 3\|u\|^2 - 2\langle u,v\rangle - 8\|v\|^2$

Later

Q2. consider the vectors $u = (2+3i, -1+5i)$, $v = (1+i, -i)$

or $v = (1-i, i)$

compute

(a) $\langle u, v\rangle$ and show that u & v are orthogonal

(b) $\|u\|$ and $\|v\|$ (c) $d(u,v)$

$u = (2+3i, -1+5i)$

$v = (1-i, i)$

$\langle u, v\rangle = (2+3i)(1-i) + (-1+5i)(i)$

Sol: (a) $\langle u, v\rangle = (2+3i)(1-i) + (-1+5i)(i)$

$= 2 - 2i + 3i + 3 - i - 5$

$= 5 + i - i - 5$

$= 0$ $\therefore$ u & v are orthogonal

(b) $\|u\| = \sqrt{(2+3i)(2-3i) + (-1+5i)(-1-5i)}$

$= \sqrt{13+26} = \sqrt{39}$

$\|v\| = \sqrt{(1+i)(1-i) + (-i)(i)}$

$= \sqrt{3}$

(c) $d(u,v) = \|u-v\| = \|(2+3i, -1+5i) - (1+i, -i)\|$

$= \|(1+2i, -1+6i)\|$

$= \sqrt{(1+2i)(1-2i) + (-1+6i)(-1-6i)} = \sqrt{5+37}$

$= \sqrt{42}$

Q3. Show that the following set is an orthonormal basi

$S = \{ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \}, \{ -\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}), (\frac{2}{3}$

$v_1 \qquad v_2 \qquad v_3$

Soln: first we show these vectors are orthogonal Then we show these vectors are normal, then we say the vectors are orthonormal.

$$v_1 \cdot v_2 = \frac{1}{\sqrt{2}} \cdot \left(\frac{-\sqrt{2}}{6}\right) + \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{6} + 0 \times \frac{2\sqrt{2}}{3}$$

$$= -\frac{1}{6} + \frac{1}{6} + 0$$

$$= 0$$

$$v_1 \cdot v_3 = \frac{1}{\sqrt{2}} \times \frac{2}{3} + \frac{1}{\sqrt{2}} \times -\frac{2}{3} + 0 \times \frac{1}{3}$$

$$= \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0$$

$$= 0$$

$$v_2 \cdot v_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Now for mal

$$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2}$$

$$= \sqrt{\frac{1}{2} + \frac{1}{2} + 0}$$

$$= \sqrt{1} = 1$$

$$\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{4}} = 1$$

$$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{\frac{4}{9} + \frac{4}{4} + \frac{1}{9}} = 1$$

Thus S is an orthonormal.

$P_3(x)$ with the inner product

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$ The standard basis

$B = \{1, x, x^2\}$ is ~~orthogonal~~, orthonormal

<u>Soln:</u> $u_1 = 1 + 0x + 0x^2$ $\quad u_2 = 0 + x + 0x^2$ $\quad u_3 = 0 + 0x + x^2$

Then

$$\langle u_1, u_2 \rangle = (1)(0) + (0)(1) + (0)(0) = 0$$

$$\langle u_1, u_3 \rangle = (1)(0) + (0)(0) + (0)(1) = 0$$

$$\langle u_2, u_3 \rangle = (0)(0) + (1)(0) + (0)(1) = 0$$

And

$$\|u_1\| = \sqrt{\langle u_1, u_1 \rangle} = \sqrt{(1)(1) + (0)(0) + (0)(0)} = 1$$

$$\|u_2\| = \sqrt{\langle u_2, u_2 \rangle} = \sqrt{(0)(0) + (1)(1) + (0)(0)} = 1$$

$$\|u_3\| = \sqrt{\langle u_3, u_3 \rangle} = \sqrt{(0)(0) + (0)(0) + (1)(1)} = 1.$$

Q4? Compute $\langle U, V \rangle$ using the inner product on $M_{2\times 2}$ where

$$U = \begin{pmatrix} 9 & -8 \\ 9 & 18 \end{pmatrix} \qquad V = \begin{pmatrix} -1 & 9 \\ 1 & 1 \end{pmatrix}$$

<u>Soln:</u>

$$\langle U, V \rangle = 9\cdot(-1) + (-8)\cdot 9 + 9\cdot 1 + 18\cdot 1$$

we are not multiply two matrix we take the inner product of two matrices

$$= -54$$

Q5. Let $\mathbb{R}^3$ have the Euclidian inner product. Find the cosine of angle θ between $u = (-1, 6, 2)$ and $v = (4, 3, -5)$

<u>Soln:</u> we know $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\|\,\|v\|}$ —(1)

So

$$\langle u, v \rangle = (-1)(4) + 6\cdot 3 + 2\times(-5) = -4 + 18 - 10 = 4$$

$$\|u\| = \sqrt{(-1)^2 + (6)^2 + 2^2} = \sqrt{1 + 36 + 4} = \sqrt{41}$$

$$\|v\| = \sqrt{4^2 + 3^2 + (-5)^2} = \sqrt{16 + 9 + 25} = \sqrt{50}$$

Put in (1)

$$\cos\theta = \frac{4}{\sqrt{41}\sqrt{50}} = \frac{4}{\sqrt{2050}}$$

Q6. let $\mathbb{R}^3$ have the Euclidean inner product. for what value of k if u & v are orthogonal
(a) $u = (2,1,3)$, $v = (1,7,k)$ (b) $u = (k,k,1)$, $v = (k,5,6)$

Sol: Sin u & v are orthogonal so
$$\langle u, v \rangle = 0$$
(a)
$$\therefore \langle (2,1,3), (1,7,k) \rangle = 0$$
$$\Rightarrow 2.1 + 1.7 + 3.k = 0$$
$$2 + 7 + 3k = 0$$
$$3k = -9$$
$$k = \frac{-9}{3} = -3$$

(b)
$$\langle (k,k,1), k, 5, 6) \rangle = 0$$
$$k^2 + 5k + 6 = 0$$
$$k^2 + 3k + 2k + 6 = 0$$
$$k(k+3) + 2(k+3) = 0$$
$$(k+2)(k+3) = 0$$
$$k = -2, -3$$

Q7. If P_2 have a usual inner product on Polynomial $p = 1 - 2x + 3x^2$, $q = 3 + x^2$ are polynomials Then find (a) $\|p\|$ (b) $\|q\|$ (c) $\langle p,q \rangle$

Sol: $\|p\| = \sqrt{1^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$

$\|q\| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$

$$\langle p, q \rangle = \langle 1 - 2x + 3x^2 \cdot (3 + x^2) \rangle$$
$$= 1.3 + (-2.0) + 3.1)$$
$$= 3 + 3 = 6$$

Theorem 6.4.2: For every system $Ax = b$, the associated normal (4) system $A^T A x = A^T b$ ...(1) is consistant and all solutions of (1) are least square solution of $Ax = b$.

Q1. Find all least square solution of the linear system

$$2x_1 = 1$$
$$-x_1 + x_2 = 0$$
$$2x_2 = \cancel{+}1$$

(b) find the error vector

Sol: we can write $Ax = b$ where

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad A^T b = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Now from the theorem: for every linear system $AX = B$, the associated normal system $A^T A X = A^T B$ —(1) is consistant and all the solutions of this normal system are least square solutions of $AX = B$.

So from (1)

$$A^T A x = A^T b \quad \text{we get}$$

$$\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Solving this system gives a unique least square solution

$$\cancel{x_1 \Rightarrow} \quad x = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} \quad \text{ie } x_1 = 1/3 \ \& \ x_2 = -1/3$$

(b) The error vector is $b - Ax$

$$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$