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Linear Programming 3: The Simplex Method

The simplex method is presented and applied to the solution of a certain class of linear programming problems.

PREREQUISITES: Gaussian elimination
Chapter 21: Linear Programming 2

INTRODUCTION

At the end of Chapter 21, we described the simplex method as a procedure for moving from one basic feasible solution of a linear programming problem to an adjacent basic feasible solution in such a way that the value of the objective function never decreases. In this chapter we shall describe the algebraic details of this procedure.

The general linear programming problem in n variables was stated as Problem 21.1 on page 304. However, in order to simplify the presentation of the simplex method, we shall restrict ourselves to linear programming problems having the following special form:

PROBLEM 22.1 *Find values of* $x_1, x_2, \ldots, x_n$ *which maximize*

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

and

$$x_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, n,$$

where

$$b_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, m.$$

In Problem 22.1, the condition that each of the m constraints be a $\leq$ inequality is not restrictive since it can easily be shown that any linear programming problem can always be written with all $\leq$ constraints. It is the condition $b_j \geq 0$ for $j = 1, 2, \ldots, m$ that is the real restriction. Nevertheless, a large class of practical problems are of this form, and the procedures developed in this chapter for Problem 22.1 are used in the application of the simplex method to the general linear programming problem.

To convert Problem 22.1 to one in standard form (see Problem 21.2 on page 304) we introduce m slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, one for each of the m constraints, to obtain

PROBLEM 22.2 *Find values of* $x_1, x_2, \ldots, x_{n+m}$ *which maximize*

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0x_{n+1} + \cdots + 0x_{n+m}$$

subject to

$$
\begin{array}{llll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} & & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + x_{n+2} & & = b_2 \\
\vdots & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & + x_{n+m} & = b_m
\end{array}
$$

and

$$x_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, n+m.$$

If we can find an optimal solution to Problem 22.2, then the values of the variables $x_1, x_2, \ldots, x_n$ will provide an optimal solution to problem 22.1.

The Simplex Tableau

In order to more clearly describe the steps in the simplex method, let us examine the following specific problem of the type we are considering in this chapter:

PROBLEM 22.3 *Find values of x_1, x_2, and x_3 which maximize*

$$z = 3x_1 - x_2 + 4x_3$$

subject to

$$
\begin{aligned}
2x_1 - x_2 + 3x_3 &\leq 5 \\
x_1 + 4x_2 - 2x_3 &\leq 1 \\
3x_1 \qquad + 6x_3 &\leq 4
\end{aligned}
$$

and

$$x_1,\ x_2,\ x_3 \geq 0.$$

To convert Problem 22.3 to one in standard form, we add slack variables x_4, x_5, x_6 to obtain

PROBLEM 22.4 *Find values of* $x_1, x_2, x_3, x_4, x_5,$ *and* x_6 *which maximize*

$$z = 3x_1 - x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$$

subject to

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + x_4 \qquad\qquad &= 5 \\ x_1 + 4x_2 - 2x_3 \qquad + x_5 \qquad &= 1 \\ 3x_1 \qquad + 6x_3 \qquad\qquad + x_6 &= 4 \end{aligned}$$

and

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

For our purposes, it will be convenient to rephrase Problem 22.4 in the following equivalent form:

PROBLEM 22.5 *Find values of* $x_1, x_2, x_3, x_4, x_5, x_6,$ *and* z *which satisfy*

$$\begin{aligned} 2x_1 - x_2 + 3x_3 + x_4 \qquad\qquad\qquad &= 5 \\ x_1 + 4x_2 - 2x_3 \qquad + x_5 \qquad\qquad &= 1 \\ 3x_1 \qquad + 6x_3 \qquad\qquad + x_6 \qquad &= 4 \\ -3x_1 + x_2 - 4x_3 \qquad\qquad\qquad + z &= 0 \end{aligned} \tag{22.1}$$

and such that $x_1, x_2, x_3, x_4, x_5, x_6$ *are nonnegative and* z *is as large as possible.*

In this formulation of the problem, z is treated as a variable on a par with x_1 through x_6, and the equation defining z in terms of the x_i is treated as an additional constraint. Thus our problem is to find a solution of the linear system of four equations in seven unknowns given by (22.1) in which one of the variables, z, is as large as possible and the other six variables are nonnegative.

The usual procedure for solving a linear system of equations is to construct the augmented matrix of the system and apply Gaussian elimination or Gauss-Jordan elimination to it to put it in row-echelon form (or reduced row-echelon form). The row-echelon form then determines the augmented matrix of an equivalent linear system which is easily solved. The simplex method proceeds along similar lines. Namely, the augmented matrix of the linear system is constructed and a variation of Gaussian elimination, called *pivotal elimination*, is applied to obtain augmented matrices in which basic feasible solutions to the linear programming problem can be determined by inspection. Let us return to Problem 22.5 to see how this is done.

The augmented matrix of the linear system (22.1) is

$$\begin{bmatrix} 2 & -1 & 3 & 1 & 0 & 0 & 0 & 5 \\ 1 & 4 & -2 & 0 & 1 & 0 & 0 & 1 \\ 3 & 0 & 6 & 0 & 0 & 1 & 0 & 4 \\ -3 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad (22.2)$$

From the way the 1's and 0's are distributed in the 4th, 5th, 6th, and 7th columns of this matrix, one particular solution of (22.1) can be seen by inspection; namely,

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 5,\ x_5 = 1,\ x_6 = 4,\ z = 0. \qquad (22.3)$$

In terms of the corresponding linear programming problem, this is

$$\mathbf{x}' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 1 \\ 4 \end{bmatrix} \quad \text{and} \quad z = 0. \qquad (22.4)$$

It is easily seen that $\mathbf{x}'$ is a basic feasible solution of linear programming Problem 22.4, according to Definition 21.5 on page 249. The three variables x_4, x_5, x_6 are the basic variables, and their corresponding values are found as the first three entries of the last column of (22.2). The value of z is the last entry of this column. That $\mathbf{x}'$ is a basic feasible solution is of crucial importance, since it is among the basic feasible solutions that we can hope to find an optimal solution. To see how we may go about finding another basic feasible solution, we rewrite (22.2) with some additional labeling:

Tableau 22.1

$$\begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & & \\ \left[\begin{array}{ccccccc|c} 2 & -1 & 3 & 1 & 0 & 0 & 0 & 5 \\ 1 & 4 & -2 & 0 & 1 & 0 & 0 & 1 \\ 3 & 0 & 6 & 0 & 0 & 1 & 0 & 4 \\ \hline -3 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} = x_4 \\ = x_5 \\ = x_6 \\ = z \end{array} \end{array}$$

Each column has been labeled with its corresponding variable from the linear system (22.1). On the right, we have labeled the entries of the last column with the corresponding variables whose values they determine in the solution given by (22.3). We have also drawn a vertical and horizontal line within the matrix in order to highlight certain entries which will be useful to us later on.

In the field of linear programming, the augmented matrix is referred to as a *tableau*. In particular, Tableau 22.1 above is called the *initial tableau* of the problem. We shall call the last row of the tableau the *objective row*, since it arises from the objective function of the original problem.

In Tableau 22.1 we have also shaded four particular columns. It can be seen that these four columns are identical to the columns of the 4×4 identity matrix. Indeed, it was exactly this fact that permitted us to find the solution given in (22.3) so easily. We had only to set those variables not corresponding to these four columns equal to zero, and then the values of the variables corresponding to the four columns were found in the last column of the augmented matrix. This suggests a way of proceeding to a new solution of the linear system. We apply appropriate elementary row operations to Tableau 22.1 to arrive at a new tableau which again contains the four columns of the 4×4 identity matrix, but this time in different position. To see how to do this, consider the following tableau, which is just Tableau 22.1 with one of its entries shaded. (We postpone for the moment a discussion of why this particular entry was chosen.)

Tableau 22.2

$$\begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & & \\ \left[\begin{array}{ccccccc|c} 2 & -1 & 3 & 1 & 0 & 0 & 0 & 5 \\ 1 & 4 & -2 & 0 & 1 & 0 & 0 & 1 \\ 3 & 0 & 6 & 0 & 0 & 1 & 0 & 4 \\ \hline -3 & 1 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right] & \begin{array}{l} = x_4 \\ = x_5 \\ = x_6 \\ = z \end{array} \end{array}$$

Our objective will be to use elementary row operations to replace the shaded entry by a "1" and obtain zeros everywhere else in that column. To do this we first divide the third row by six to obtain:

Tableau 22.3

$$\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & \\
\hline
2 & -1 & 3 & 1 & 0 & 0 & 0 & 5 \\
1 & 4 & -2 & 0 & 1 & 0 & 0 & 1 \\
1/2 & 0 & \boxed{1} & 0 & 0 & 1/6 & 0 & 2/3 \\
\hline
-3 & 1 & -4 & 0 & 0 & 0 & 1 & 0
\end{array}$$

Next we perform the following three elementary row operations:

1. Add -3 times the 3rd row to the 1st row.
2. Add 2 times the 3rd row to the 2nd row.
3. Add 4 times the 3rd row to the 4th row.

The result is the following tableau:

Tableau 22.4

$$\begin{array}{ccccccc|cl}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & & \\
\hline
1/2 & -1 & 0 & 1 & 0 & -1/2 & 0 & 3 & = x_4 \\
2 & 4 & 0 & 0 & 1 & 1/3 & 0 & 7/3 & = x_5 \\
1/2 & 0 & 1 & 0 & 0 & 1/6 & 0 & 2/3 & = x_3 \\
\hline
-1 & 1 & 0 & 0 & 0 & 2/3 & 1 & 8/3 & = z
\end{array}$$

It can be seen that Tableau 22.4 contains within it the four columns of the 4×4 identity matrix, though not in their usual order. Consequently, if we set the variables not associated with these columns equal to zero, we obtain the following solution to linear system (22.1):

$$x_1 = 0,\ x_2 = 0,\ x_3 = 2/3,\ x_4 = 3,\ x_5 = 7/3,\ x_6 = 0,\ z = 8/3. \qquad (22.5)$$

As before, we have labeled the entries in the last column with the variables whose values they determine in this solution.

Solution (22.5) specifies the following vector solution of Problem 22.4:

$$\mathbf{x}'' = \begin{bmatrix} 0 \\ 0 \\ 2/3 \\ 3 \\ 7/3 \\ 0 \end{bmatrix} \quad \text{and} \quad z = 8/3. \tag{22.6}$$

We see that x'' is a basic feasible solution with basic variables x_3, x_4, x_5. We also see that x' as given in (22.4) and x'' are *adjacent basic feasible solutions* since they have $m - 1 = 2$ basic variables in common; namely, x_4 and x_5. That is, the elementary row operations we performed on Tableau 22.1 to obtain Tableau 22.4 took us from the basic feasible solution x' to the adjacent basic feasible solution x''. In addition, the value of the objective function increased from $z = 0$ to $z = 8/3$.

The shaded entry in Tableau 22.2 is called the *pivot entry* of that particular tableau. The column in which it lies is called the *pivot column*, and the row in which it lies is called the *pivot row*. After the elementary row operations were applied to obtain Tableau 22.4, the pivot column contained all zeros, except for a one in the pivot entry position. Thus the pivot column was converted to one of the columns of the 4×4 identity matrix, and the variable x_3 corresponding to the pivot column was converted from a nonbasic variable to a basic variable. At the same time, the variable x_6 which labeled the pivot row, was converted from a basic variable to a nonbasic variable. That is, x_3 replaced x_6 as a basic variable in going from Tableau 22.1 to Tableau 22.4. For this reason, x_3 is called the *entering variable* and x_6 is called the *departing variable* of Tableau 22.1.

Let us see if we can increase the value of the objective function above the value $z = 8/3$ attained in Tableau 22.4. Below we have rewritten Tableau 22.4 with a pivot entry shaded. Again, we postpone until the next section a discussion of how this pivot entry was chosen.

Tableau 22.5

$$\begin{array}{r|ccccccc|c|l}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & & \\
\hline
 & 1/2 & -1 & 0 & 1 & 0 & -1/2 & 0 & 3 & = x_4 \\
\text{Depart } x_5 \leftarrow & \boxed{2} & 4 & 0 & 0 & 1 & 1/3 & 0 & 7/3 & = x_5 \\
 & 1/2 & 0 & 1 & 0 & 0 & 1/6 & 0 & 2/3 & = x_3 \\
\hline
 & -1 & 1 & 0 & 0 & 0 & 2/3 & 1 & 8/3 & z \\
\hline
 & \uparrow & & & & & & & & \\
 & \text{Enter } x_1 & & & & & & & &
\end{array}$$

We apply the elementary row operations to the first column to convert it to a column with a "1" in the pivot position and zeros everywhere else. Thus, x_1 will be the entering variable and x_5 will be the departing variable, as the arrows in the tableau indicate. The reader can easily verify that the elementary row operations necessary will produce the next tableau:

Tableau 22.6

x_1	x_2	x_3	x_4	x_5	x_6	z		
0	-2	0	1	-1/4	-7/12	0	29/12	$= x_4$
1	2	0	0	1/2	1/6	0	7/6	$= x_1$
0	-1	1	0	-1/4	1/12	0	1/12	$= x_3$
0	3	0	0	1/2	5/6	1	23/6	$= z$

Setting those variables not corresponding to the shaded columns equal to zero yields the solution

$$x_1 = 7/6,\ x_2 = 0,\ x_3 = 1/12,\ x_4 = 29/12,\ x_5 = 0,\ x_6 = 0,\ z = 23/6. \tag{22.7}$$

For the linear programming problem, we then have the following basic feasible solution and objective function value:

$$\mathbf{x}''' = \begin{bmatrix} 7/6 \\ 0 \\ 1/12 \\ 29/12 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad z = 23/6. \tag{22.8}$$

As we shall show in the next section, the value $z = 23/6$ is the largest value the objective function can assume over the feasible set. Thus we have reached an optimal solution. For the solution to Problem 22.3, we discard the slack variables x_4, x_5, x_6 and write

$$x_1 = 7/6, \quad x_2 = 0, \quad x_3 = 1/12 \tag{22.9}$$

as the optimal solution, with the corresponding maximum $z = 23/6$ for the objective function.

This example illustrates the kinds of calculations required to implement the simplex method. In the next section we shall discuss how to choose the pivot entry and how to determine if an optimal solution has been reached.

Steps in the Simplex Method

In this section we shall outline the steps in the simplex method and give an example. In the next section we shall discuss their mathematical justification. The simplex method consists of the following five steps:

STEP 1 *Construct the initial tableau.*

STEP 2 *Test for optimality. If the tableau yields an optimal solution, then stop; otherwise, continue to Step 3.*

STEP 3 *Determine the pivot column.*

STEP 4 *Determine the pivot row.*

STEP 5 *Apply the elementary row operations to obtain all zeros in the pivot column, except for a "one" in the pivot row. Return to Step 2.*

The details for Steps 2, 3, and 4 are as follows:

Test for Optimality. If all of the entries in the objective row are nonnegative (ignoring the rightmost entry) the tableau determines an optimal solution.

Determination of Pivot Column. Choose the pivot column so that it contains the most negative entry of the objective row (ignoring the rightmost entry).

Determination of Pivot Row. Ignoring the objective row, divide each positive entry of the pivot column into the last entry in its row. Choose the pivot row to be one which yields the smallest such ratio.

The reader should return to the tableaus constructed in connection with Problem 22.5 to verify that the pivot entries in each tableau were selected according to the above rules, and also to verify that Tableau 22.6 determines an optimal solution to the problem.

Let us apply the simplex method as described above to the following example:

Example 22.1 Find values of x_1, x_2, x_3 which maximize

$$z = 3x_1 + 4x_2 + 2x_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &\leq 15 \\ x_1 + 2x_2 + 3x_3 &\leq 7 \\ 2x_1 + x_2 + x_3 &\leq 6 \end{aligned}$$

and

$$x_1, x_2, x_3 \geq 0.$$

SOLUTION In standard form, this problem is

$$\text{Maximize } z = 3x_1 + 4x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 + x_4 \qquad\qquad &= 15 \\ x_1 + 2x_2 + 3x_3 \qquad + x_5 \qquad &= 7 \\ 2x_1 + x_2 + x_3 \qquad\qquad + x_6 &= 6 \end{aligned}$$

and

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The initial tableau for the problem is then

Tableau 22.7

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & \end{matrix} \\ \left[\begin{array}{ccccccc|c} 3 & 2 & 4 & 1 & 0 & 0 & 0 & 15 \\ 1 & 2 & 3 & 0 & 1 & 0 & 0 & 7 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 & 6 \\ \hline -3 & -4 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{matrix} = x_4 \\ = x_5 \\ = x_6 \\ = z \end{matrix} \end{array}$$

The objective row contains negative entries, so that the initial tableau does not determine an optimal solution. The most negative entry, -4, lies in the second column, so that the second column will be the pivot column. To determine the pivot row, we evaluate the following ratios:

1st row: $15/2 = 7\frac{1}{2}$.   2nd row: $7/2 = 3\frac{1}{2}$.   3rd row: $6/1 = 6$.

The 2nd row yields the smallest ratio so that it will be the pivot row. So far we have the following:

Tableau 22.8

	x_1	x_2	x_3	x_4	x_5	x_6	z		
Depart	3	2	4	1	0	0	0	15	$= x_4$
x_5 ←	1	$\boxed{2}$	3	0	1	0	0	7	$= x_5$
	2	1	1	0	0	1	0	6	$= x_6$
	-3	-4	-2	0	0	0	1	0	$= z$
		↑ Enter x_2							

We now apply the following elementary row operations to Tableau 22.8:

1. Divide the 2nd row by 2.
2. Add -2 times the 2nd row to the 1st row.
3. Add -1 times the 2nd row to the 3rd row.
4. Add 4 times the 2nd row to the 4th row.

The resulting tableau is the following:

Tableau 22.9

x_1	x_2	x_3	x_4	x_5	x_6	z		
2	0	1	1	-1	0	0	8	$= x_4$
1/2	1	3/2	0	1/2	0	0	7/2	$= x_2$
3/2	0	-1/2	0	-1/2	1	0	5/2	$= x_6$
-1	0	4	0	2	0	1	14	$= z$

The objective row still contains a negative entry, so that we have not yet reached an optimal solution. The new pivot column is the first since it contains the only negative entry of the objective row. To determine the pivot row, we evaluate the following ratios:

1st row: $8/2 = 4$. 2nd row: $(7/2)/(1/2) = 7$. 3rd row: $(5/2)/(3/2) = 1\frac{2}{3}$.

The third row yields the smallest ratio, so that it is the new pivot row. So far, we have

Tableau 22.10

$$\begin{array}{c|c}
 & \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & \end{array} \\
\begin{array}{r} \\ \text{Depart} \\ x_6 \leftarrow \\ \\ \end{array} & \left[\begin{array}{ccccccc|c}
2 & 0 & 1 & 1 & -1 & 0 & 0 & 8 \\
1/2 & 1 & 3/2 & 0 & 1/2 & 0 & 0 & 7/2 \\
\boxed{3/2} & 0 & -1/2 & 0 & -1/2 & 1 & 0 & 5/2 \\
\hline
-1 & 0 & 4 & 0 & 2 & 0 & 1 & 14
\end{array}\right] \begin{array}{l} = x_4 \\ = x_2 \\ = x_6 \\ = z \end{array}
\end{array}$$

$\uparrow$ Enter x_1

We now apply the following elementary row operations to Tableau 22.10:

1. Divide the 3rd row by 3/2.
2. Add -2 times the 3rd row to the 1st row.
3. Add -½ times the 3rd row to the 2nd row.
4. Add 1 times the 3rd row to the 4th row.

The resulting tableau is

Tableau 22.11

$$\begin{array}{ccccccc|cl}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & & \\
0 & 0 & 5/3 & 1 & -1/3 & -4/3 & 0 & 14/3 & = x_4 \\
0 & 1 & 5/3 & 0 & 2/3 & -1/3 & 0 & 8/3 & = x_2 \\
1 & 0 & -1/3 & 0 & -1/3 & 2/3 & 0 & 5/3 & = x_1 \\
\hline
0 & 0 & 11/3 & 0 & 5/3 & 2/3 & 1 & 47/3 & = z
\end{array}$$

The objective row of this tableau does not contain any negative entries and so this tableau determines an optimal solution. The basic variables in the optimal basic feasible solution are x_1, x_2, and x_4 as the righthand labeling indicates. The optimal solution is

$$x_1 = 5/3,\ x_2 = 8/3,\ x_3 = 0,\ x_4 = 14/3,\ x_5 = 0,\ x_6 = 0,\ z = 47/3.$$

For the original problem posed in this example, we discard the slack variables x_4, x_5, x_6 and simply write

$$x_1 = 5/3, \quad x_2 = 8/3, \quad x_3 = 0, \quad z = 47/3.$$

We conclude this section with some remarks concerning complications which may arise in the use of the simplex method as we have described it:

1. In Step 3 it is possible that there is a tie for the most negative entry in the objective row. In that case, any one of them may be chosen, and no complications arise.
2. In Step 4 it is possible that there is more than one row with the same smallest ratio. In that case, any one of them may be used to determine the pivot row and no complications arise in the calculations. However, if such a tie arises, it can be shown that the basic feasible solution determined by the next tableau will be degenerate (i.e., will have a basic variable whose value is zero). As discussed in the last chapter, it is degeneracy which may bring about cycling. But as we also mentioned, it is more a theoretical problem than a practical problem.
3. In Step 4 it is possible that no entry in the pivot column is positive, in which case our technique for evaluating the pivot row is meaningless. It can be shown that if this situation arises, the problem has an *unbounded solution*.

Justification of the Steps in the Simplex Method

Let us return to the linear programming problem in $n+m$ variables posed in Problem 22.2. Suppose at some point in our calculations we have arrived at Tableau 22.12. (In Exercise 22.11, we ask the reader to show that in any tableau the column labeled "z" always has the form indicated.) Thus the current basic variables are $x_{B1}, x_{B2}, \ldots, x_{Bm}$ with corresponding values $d_1, d_2, \ldots, d_m$, and the current value of the objective variable is w. Let us see if the entry y_{rs} would make a suitable pivot entry. The entering variable would be x_s and the departing variable would be x_{Br}. If $y_{rs} \neq 0$, the elementary row operations in Step 5 of the simplex method will produce a tableau having the form of Tableau 22.13.

Tableau 22.12

$$
\begin{array}{rcl}
 & \begin{array}{ccccccc} x_1 & x_2 & \cdots & x_s & \cdots & x_{n+m} & z \end{array} & \\
\begin{array}{r} \\ \\ \\ \text{Depart} \\ x_{Br} \leftarrow \\ \\ \\ \\ \end{array} &
\left[\begin{array}{ccccccc|c}
y_{11} & y_{12} & \cdots & y_{1s} & \cdots & y_{1,n+m} & 0 & d_1 \\
y_{21} & y_{22} & \cdots & y_{2s} & \cdots & y_{2,n+m} & 0 & d_2 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
y_{r1} & y_{r2} & \cdots & \boxed{y_{rs}} & \cdots & y_{r,n+m} & 0 & d_r \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
y_{m1} & y_{m2} & \cdots & y_{ms} & \cdots & y_{m,n+m} & 0 & d_m \\
\hline
c_1 & c_2 & \cdots & c_s & \cdots & c_{n+m} & 1 & w
\end{array}\right] &
\begin{array}{l} = x_{B1} \\ = x_{B2} \\ \vdots \\ = x_{Br} \\ \vdots \\ = x_{Bm} \\ = z \end{array} \\
 & \begin{array}{c} \uparrow \\ \text{Enter } x_s \end{array} &
\end{array}
$$

Tableau 22.13

$$
\begin{array}{cl}
\begin{array}{ccccccc} x_1 & x_2 & \cdots & x_s & \cdots & x_{n+m} & z \end{array} & \\
\left[\begin{array}{ccccccc|c}
y^*_{11} & y^*_{12} & \cdots & 0 & \cdots & y^*_{1,n+m} & 0 & d_1 - y_{1s}d_r/y_{rs} \\
y^*_{21} & y^*_{22} & \cdots & 0 & \cdots & y^*_{2,n+m} & 0 & d_2 - y_{2s}d_r/y_{rs} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
y^*_{r1} & y^*_{r2} & \cdots & 1 & \cdots & y^*_{r,n+m} & 0 & d_r/y_{rs} \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\
y^*_{m1} & y^*_{m2} & \cdots & 0 & \cdots & y^*_{m,n+m} & 0 & d_m - y_{ms}d_r/y_{rs} \\
\hline
c^*_1 & c^*_2 & \cdots & 0 & \cdots & c^*_{n+m} & 1 & w - c_s d_r/y_{rs}
\end{array}\right] &
\begin{array}{l} = x_{B1} \\ = x_{B2} \\ \vdots \\ = x_s \\ \vdots \\ = x_{Bm} \\ = z \end{array}
\end{array}
$$

The pivot column now contains all zeros except for the "1" in the previous pivot entry position. All of the other entries in the tableau have new values, which we indicate with asterisks, except in the last column where we have explicitly written the new values in terms of the entries in Tableau 22.12. First, let us examine the value of the entering variable x_s:

$$x_s = d_r / y_{rs} \,. \tag{22.10}$$

Since $d_r \geq 0$, we see that we must have $y_{rs} > 0$ in order to satisfy the constraint $x_s \geq 0$. Let us list this fact as

OBSERVATION 22.1 *The pivot entry must be positive in order that the new tableau determine a feasible solution.*

The remaining basic variables have values given by

$$x_{Bi} = d_i - y_{is} d_r / y_{rs} \qquad \text{for} \quad i = 1, 2, \ldots, m;\ i \neq s \,. \tag{22.11}$$

We must have $x_{Bi} \geq 0$ for the new solution to be feasible. Now if for some i we have $y_{is} \leq 0$, then (22.11) states that for that i, $x_{Bi} \geq 0$ since $d_i \geq 0$, $d_r \geq 0$, and $y_{rs} > 0$. On the other hand, if for some i we have $y_{is} > 0$, then (21.11) requires that

$$d_i - y_{is} d_r / y_{rs} \geq 0 \tag{22.12}$$

in order that $x_{Bi} \geq 0$ for that i. Equation (22.12) can also be written as

$$\frac{d_r}{y_{rs}} \leq \frac{d_i}{y_{is}} \,. \tag{22.13}$$

In other words, Eq. (22.13) must be satisfied for all those i for which $y_{is} > 0$ in order that the new tableau determine a feasible solution. We state this as

OBSERVATION 22.2 *In order that the new tableau determine a feasible solution, the following must be true: The ratio of the element in the rightmost column of the pivot row to the pivot entry must be the smallest of the corresponding ratios in all of the other rows which contain positive entries in the pivot column (ignoring the objective row).*

Next, let us examine the new value w^* of the objective function. From Tableau 22.13, we see that

$$w^* = w - c_s d_r / y_{rs} . \tag{22.14}$$

Ideally, we would want the increase in the objective function

$$w^* - w = -c_s d_r / y_{rs} \tag{22.15}$$

to be as large as possible. But this would require that we compute the quantities

$$-c_s d_r / y_{rs}$$

for all possible values of r and s to find the largest one. Usually this is not done because of the large number of calculations this would require. Instead, the entering variable x_s is chosen so that c_s is as negative as possible. Since $d_r \geq 0$ and $y_{rs} > 0$, Eq. (22.15) then guarantees that

$$w^* - w \geq 0 ;$$

i.e., that the objective function does not decrease. As discussed in the last chapter, this always eventually leads to an optimal solution, except for the very remote possibility of cycling. Let us summarize this as follows:

OBSERVATION 22.3 *Choose the pivot column so that it contains the most negative entry of the objective row (ignoring the entry in the rightmost column).*

Equation (22.15) also tells us the following: If all of the c's are nonnegative then the value of the objective function cannot increase, regardless of the choice of pivot entry. In this case, we must have already attained the maximum value of the objective function. Thus, we have

OBSERVATION 22.4 *If all of the entries in the objective row, except for the rightmost entry, are nonnegative, the tableau determines an optimal solution.*

The above four observations justify the steps in the simplex method.

Readers interested in pursuing linear programming in more detail are referred to the following texts:

S. I. Gass, *Linear Programming*, 4th ed. New York: McGraw-Hill Book Company, 1975.

L. Cooper and D. Steinberg, *Methods and Applications of Linear Programming*, Philadelphia: W. B. Saunders Company, 1974.

EXERCISES

In Exercises 22.1 to 22.6 solve the given linear programming problem by the simplex method.

22.1 Maximize $z = 3x_1 + 4x_2$

subject to

$$2x_1 + 3x_2 \leq 7$$
$$5x_1 + 2x_2 \leq 3$$

and

$$x_1, x_2 \geq 0.$$

22.2 Maximize $z = 2x_1 + x_2$

subject to

$$3x_1 + 2x_2 \leq 4$$
$$3x_1 + x_2 \leq 3$$
$$2x_1 \leq 3$$

and

$$x_1, x_2 \geq 0.$$

22.3 Maximize $z = 3x_1 - 2x_2 + 6x_3$

subject to

$$2x_1 - 5x_2 + x_3 \leq 2$$
$$x_1 + x_2 + x_3 \leq 5$$

and

$$x_1, x_2, x_3 \geq 0.$$

22.4 Maximize $z = 2x_1 + x_2 - x_3$

subject to

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &\le 2 \\ x_1 + 5x_2 - 2x_3 &\le 4 \\ 2x_1 - 4x_2 - x_3 &\le 3 \end{aligned}$$

and

$$x_1,\ x_2,\ x_3 \ge 0.$$

22.5 Maximize $z = 3x_1 - 2x_2 - x_3 + x_4$

subject to

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - x_4 &\le 6 \\ x_1 + 2x_2 - x_3 + 2x_4 &\le 4 \end{aligned}$$

and

$$x_1,\ x_2,\ x_3,\ x_4 \ge 0$$

22.6 Maximize $z = x_1 - 2x_2 + 3x_3 + x_4$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 + 3x_4 &\le 8 \\ 2x_1 + 3x_2 - x_3 + 2x_4 &\le 5 \\ x_1 + x_2 - 3x_3 + 4x_4 &\le 6 \end{aligned}$$

and

$$x_1,\ x_2,\ x_3,\ x_4 \ge 0$$

22.7 Solve Example 20.1 by the simplex method.

22.8 Solve Exercise 20.1 by the simplex method.

22.9 Solve Exercise 20.6 by the simplex method.

22.10 Solve Exercise 20.7 by the simplex method.

22.11 Show that in any tableau the column labeled "z" always has the form indicated in Tableau 22.12.