20

Linear Programming 1: A Geometric Approach

A geometric technique for maximizing or minimizing a linear expression in two variables subject to a set of linear constraints is described.

PREREQUISITES:	Linear systems Linear inequalities

INTRODUCTION

The study of linear programming theory has expanded greatly since the pioneer work of George Dantzig in the late nineteen-forties. Today, linear programming is applied to a wide variety of problems in industry and science. In this chapter we present a geometric approach to the solution of simple linear programming problems. In Chapters 21 and 22 we develop the algebraic theory required to solve more general problems in this field.

Let us begin with some examples:

EXAMPLE 20.1 A candy manufacturer has 130 pounds of chocolate-covered cherries and 170 pounds of chocolate-covered mints in stock. He decides to sell them in the form of two different mixtures. One mixture will contain half cherries and half mints and will sell for \$2.00 per pound. The other mixture will contain one-third cherries and two-thirds mints and will sell for \$1.25 per pound. How many pounds of each mixture should the candy manufacturer prepare in order to maximize his sales revenue?

Let us first formulate this problem mathematically. Let the mixture of half cherries and half mints be called mix A, and let x_1 be the number of pounds of this mixture to be prepared. Let the mixture of one-third cherries and two-thirds mints be called mix B, and let x_2 be the number of pounds of this mixture to be prepared. Since mix A sells for \$2.00 per pound and mix B sells for \$1.25 per pound, the total sales z (in dollars) will be

$$z = 2.00x_1 + 1.25x_2.$$

Since each pound of mix A contains 1/2 pound of cherries and each pound of mix B contains 1/3 pound of cherries, the total number of pounds of cherries used in both mixtures is

$$\frac{1}{2}x_1 + \frac{1}{3}x_2.$$

Similarly, since each pound of mix A contains 1/2 pound of mints and each pound of mix B contains 2/3 pound of mints, the total number of pounds of mints used in both mixtures is

$$\frac{1}{2}x_1 + \frac{2}{3}x_2.$$

Because the manufacturer can use at most 130 pounds of cherries and 170 pounds of mints, we must have

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 \leq 130$$

$$\frac{1}{2}x_1 + \frac{2}{3}x_2 \leq 170.$$

Also, since x_1 and x_2 cannot be negative numbers, we must have

$$x_1 \geq 0 \qquad \text{and} \qquad x_2 \geq 0.$$

The problem can therefore be formulated mathematically as follows:

Find values of x_1 and x_2 which maximize

$$z = 2.00x_1 + 1.25x_2$$

subject to

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 \le 130$$

$$\frac{1}{2}x_1 + \frac{2}{3}x_2 \le 170$$

$$x_1 \ge 0$$

$$x_2 \ge 0.$$

In the next section we shall show how to solve this type of mathematical problem geometrically.

EXAMPLE 20.2 A woman has up to \$10,000 to invest. Her broker suggests investing in two bonds, *A* and *B*. Bond *A* is a rather risky bond with an annual yield of 10%, and bond *B* is a rather safe bond with an annual yield of 7%. After some consideration, she decides to invest at most \$6,000 in bond *A*, at least \$2,000 in bond *B*, and to invest at least as much in bond *A* as in bond *B*. How should she invest her \$10,000 in order to maximize her annual yield?

In order to formulate this problem mathematically, let x_1 be the number of dollars to be invested in bond *A* and let x_2 be the number of dollars to be invested in bond *B*. Since each dollar invested in bond *A* earns \$.10 per year and each dollar invested in bond *B* earns \$.07 per year, the total dollar amount z earned each year by both bonds is

$$z = .10x_1 + .07x_2.$$

The constraints imposed can be formulated mathematically as follows:

Invest no more than \$10,000:	$x_1 + x_2 \le 10{,}000$
Invest at most \$6,000 in bond *A*:	$x_1 \le 6{,}000$
Invest at least \$2,000 in bond *B*:	$x_2 \ge 2{,}000$
Invest at least as much in bond *A* as in bond *B*:	$x_1 \ge x_2.$

We also have the implicit assumption that x_1 and x_2 are nonnegative:

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0.$$

Thus, the complete mathematical formulation of the problem is as follows:

Find values of x_1 *and* x_2 *which maximize*

$$z = .10x_1 + .07x_2$$

subject to

$$x_1 + x_2 \le 10{,}000$$
$$x_1 \le 6{,}000$$
$$x_2 \ge 2{,}000$$
$$x_1 - x_2 \ge 0$$
$$x_1 \ge 0$$
$$x_2 \ge 0.$$

EXAMPLE 20.3 A student desires to design a breakfast of corn flakes and milk which is as economical as possible. On the basis of what he eats during his other meals, he decides that his breakfast should supply him with at least nine grams of protein, at least one-third the recommended daily allowance (RDA) of vitamin D, and at least one-fourth the RDA of calcium. He finds the following nutrition information on the milk and corn flakes containers:

	Milk ½ cup	Corn Flakes 1 ounce
Cost	7.5¢	5.0¢
Protein	4 grams	2 grams
Vitamin D	1/8 of RDA	1/10 of RDA
Calcium	1/6 of RDA	none

In order not to have his mixture too soggy or too dry, the student decides to limit himself to mixtures which contain one to three ounces of corn flakes per cup of milk, inclusive. What quantities of milk and corn flakes should he use to minimize the cost of his breakfast?

For the mathematical formulation of this problem, let x_1 be the quantity of milk used measured in ½-cup units and let x_2 be the quantity of corn flakes used measured in 1-ounce units. Then if z is the cost of the breakfast in cents, we may write the following:

Cost of breakfast: $z = 7.5x_1 + 5.0x_2$

At least nine grams of protein: $4x_1 + 2x_2 \ge 9$

At least 1/3 of RDA of vitamin D: $\frac{1}{8}x_1 + \frac{1}{10}x_2 \ge \frac{1}{3}$

At least 1/4 of RDA of calcium: $\frac{1}{6}x_1 \geq \frac{1}{4}$

At least one ounce of corn flakes per cup (two ½-cups) of milk: $x_2/x_1 \geq \frac{1}{2}$ (or $x_1 - 2x_2 \leq 0$)

At most three ounces of corn flakes per cup (two ½-cups) of milk: $x_2/x_1 \leq \frac{3}{2}$ (or $3x_1 - 2x_2 \geq 0$)

As before, we also have the implicit assumption that $x_1 \geq 0$ and $x_2 \geq 0$. Thus the complete mathematical formulation of the problem is as follows:

Find values of x_1 and x_2 which minimize

$$z = 7.5x_1 + 5.0x_2$$

subject to

$$4x_1 + 2x_2 \geq 9$$

$$\frac{1}{8}x_1 + \frac{1}{10}x_2 \geq \frac{1}{3}$$

$$\frac{1}{6}x_1 \geq \frac{1}{4}$$

$$x_1 - 2x_2 \leq 0$$

$$3x_1 - 2x_2 \geq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$

Geometric Solution of Linear Programming Problems

Each of the three examples in the introduction is a special case of the following problem:

PROBLEM 20.1 *Find values of* x_1 *and* x_2 *which either maximize or minimize*

$$z = c_1x_2 + c_2x_2 \tag{20.1}$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \ (\leq)(\geq)(=) \ b_1 \\ a_{21}x_1 + a_{22}x_2 \ (\leq)(\geq)(=) \ b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \ (\leq)(\geq)(=) \ b_m \end{aligned} \tag{20.2}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0. \tag{20.3}$$

In each of the m equations of (20.2), any one of the symbols $\leq$, $\geq$, $=$ may be used.

Problem 17.1 is the *general linear programming problem* in two variables. The linear function z in (20.1) is called the *objective function*. Equations (20.2) and (20.3) are called the *constraints*; in particular, Eqs. (20.3) are called the *nonnegativity constraints* on the variables x_1 and x_2.

We shall now show how to solve a linear programming problem in two variables graphically. A pair of values (x_1, x_2) which satisfy all of the constraints is called a *feasible solution*. The set of all feasible solutions determines a subset of the x_1x_2-plane called the *feasible region*. Our desire is to find a feasible solution which maximizes the objective function. Such a solution is called an *optimal solution*.

To examine the feasible region of a linear programming problem, let us note that each constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 = b_i$$

defines a line in the x_1x_2-plane, while each constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 \leq b_i$$

or

$$a_{i1}x_1 + a_{i2}x_2 \geq b_i$$

defines a half-plane which includes its boundary line

$$a_{i1}x_1 + a_{i2}x_2 = b_i.$$

Thus, the feasible region is always an intersection of finitely many lines and half-planes. For example, the four constraints

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 \leq 130$$

$$\frac{1}{2}x_1 + \frac{2}{3}x_2 \leq 170$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

of Example 20.1 define the half-planes illustrated in Fig. 20.1(a), (b), (c), and (d). The feasible region of this problem is thus the intersection of these four half-planes, which is illustrated in Fig. 20.1(e).

It can be shown that the feasible region of a linear programming problem has a boundary consisting of a finite number of straight-line segments. If the feasible region can be enclosed in a sufficiently large circle, it is called *bounded* (Fig. 20.1(e)); otherwise it is called *unbounded* (Fig. 20.5). If the feasible region is *empty* (contains no points), then the constraints are inconsistent and the linear programming problem has no solution (Fig. 20.6).

Those boundary points of a feasible region which are intersections of two of the straight-line boundary segments are called *extreme points*. (They are also called *corner points* or *vertex points*.) For example, from Fig. 20.1(e), the feasible region of Example 20.1 has four extreme points:

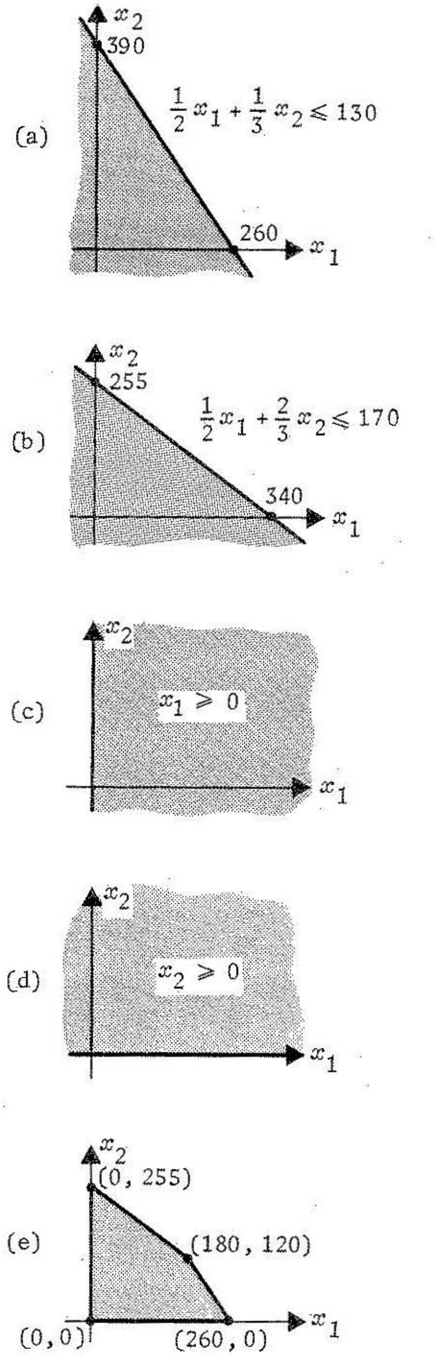

Figure 20.1

$$(0, 0), \quad (0, 255), \quad (180, 120), \quad (260, 0) . \qquad (20.4)$$

The importance of the extreme points of a feasible region is shown by the following theorem:

THEOREM 20.1 *If the feasible region of a linear programming problem is nonempty and bounded, then the objective function attains both a maximum and minimum value and these occur at extreme points of the feasible region. If the feasible region is unbounded, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at an extreme point.*

Figure 20.2 suggests the idea behind the proof of this theorem. Since the objective function

$$z = c_1x_1 + c_2x_2$$

of a linear programming problem is a linear function of x_1 and x_2, its level curves (the curves along which z has constant values) are straight lines. As we move in a direction perpendicular to these level curves, the objective function either increases or decreases monotonically. Within a bounded feasible region, the maximum and minimum values of z must therefore occur at extreme points, as Figure 20.2 indicates.

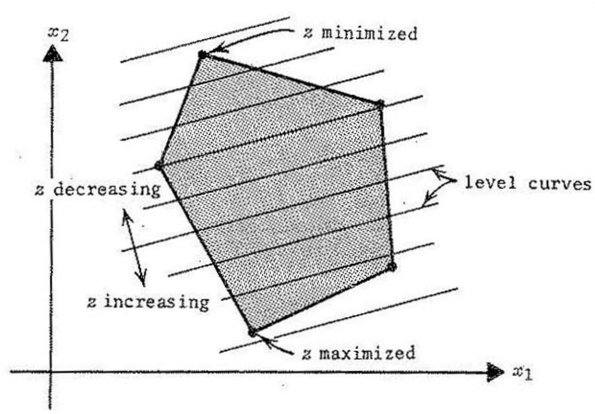

Figure 20.2

In the next few examples, we use Theorem 20.1 to solve several linear programming problems and illustrate the variations in the nature of the solutions which may occur.

EXAMPLE 20.1 (REVISITED) From Figure 20.1(e), we see that the feasible region of Example 20.1 is bounded. Consequently, from Theorem 20.1 the objective function

$$z = 2.00x_1 + 1.25x_2$$

attains both its minimum and maximum values at extreme points. The four extreme points and the corresponding values of z are given in the following table

Extreme point (x_1, x_2)	Value of $z = 2.00x_1 + 1.25x_2$
(0, 0)	0
(0, 255)	318.75
(180, 120)	510.00
(260, 0)	520.00

We see that the largest value of z is 520.00, and the corresponding optimal solution is (260, 0). Thus, the candy manufacturer attains maximum sales of $520 when he produces 260 pounds of mixture A and none of mixture B.

EXAMPLE 20.4 Find values of x_1 and x_2 which maximize

$$z = x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 24 \\ x_1 - x_2 &\leq 7 \\ x_2 &\leq 6 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

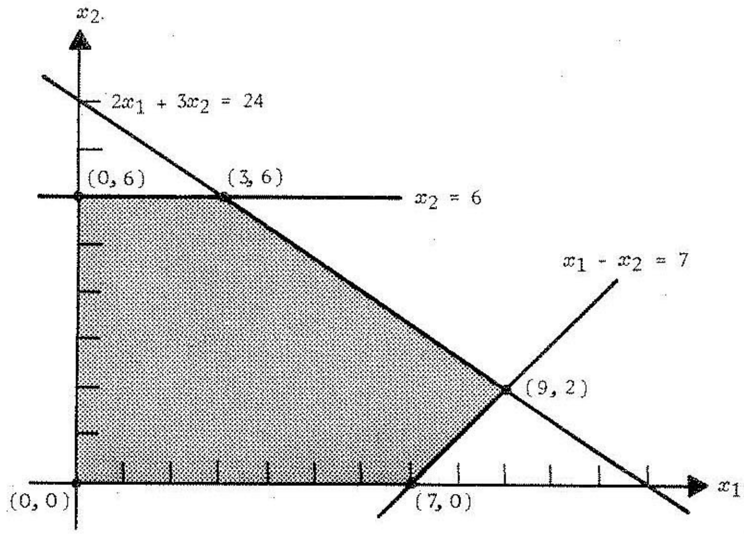

Figure 20.3

SOLUTION In Fig. 20.3 we have drawn the feasible region of this problem. Since it is bounded, the maximum value of z is attained at one of the five extreme points. The values of the objective function at the five extreme points are given in the following table:

Extreme point (x_1, x_2)	Value of $z = x_1 + 3x_2$
(0, 6)	18
(3, 6)	21
(9, 2)	15
(7, 0)	7
(0, 0)	0

From this table, the maximum value of z is 21, which is attained at $x_1 = 3$ and $x_2 = 6$.

EXAMPLE 20.5 Find values of x_1 and x_2 which maximize

$$z = 4x_1 + 6x_2$$

subject to

$$2x_1 + 3x_2 \leq 24$$
$$x_1 - x_2 \leq 7$$
$$x_2 \leq 6$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

SOLUTION The constraints in this problem are identical to the constraints in Example 20.4, and so the feasible region of this problem is also given by Fig. 20.3. The values of the objective function at the extreme points are as follows:

Extreme point (x_1, x_2)	Value of $z = 4x_1 + 6x_2$
(0, 6)	36
(3, 6)	48
(9, 2)	48
(7, 0)	28
(0, 0)	0

We see that the objective function attains a maximum value of 48 at two adjacent extreme points, (3, 6) and (9, 2). This shows that an optimal solution to a linear programming problem need not be unique. As we ask the reader to show in Exercise 20.9, if the objective function has the same value at two adjacent extreme points, it has the same value at all points on the straight-line boundary segment connecting the two extreme points. Thus, in this example the maximum value of z is attained at all points on the straight-line segment connecting the extreme points (3, 6) and (9, 2).

EXAMPLE 20.6 Find values of x_1 and x_2 which minimize

$$z = 2x_1 - x_2$$

subject to

$$2x_1 + 3x_2 = 12$$
$$2x_1 - 3x_2 \geq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

SOLUTION In Fig. 20.4 we have drawn the feasible region of this problem. Because one of the constraints is an equality constraint, the feasible region is a straight line segment with two extreme points. The values of z at the two extreme points are as follows:

Extreme point (x_1, x_2)	Value of $z = 2x_1 - x_2$
(3, 2)	4
(6, 0)	12

The minimum value of z is thus 4 and is attained at $x_1 = 3$ and $x_2 = 2$.

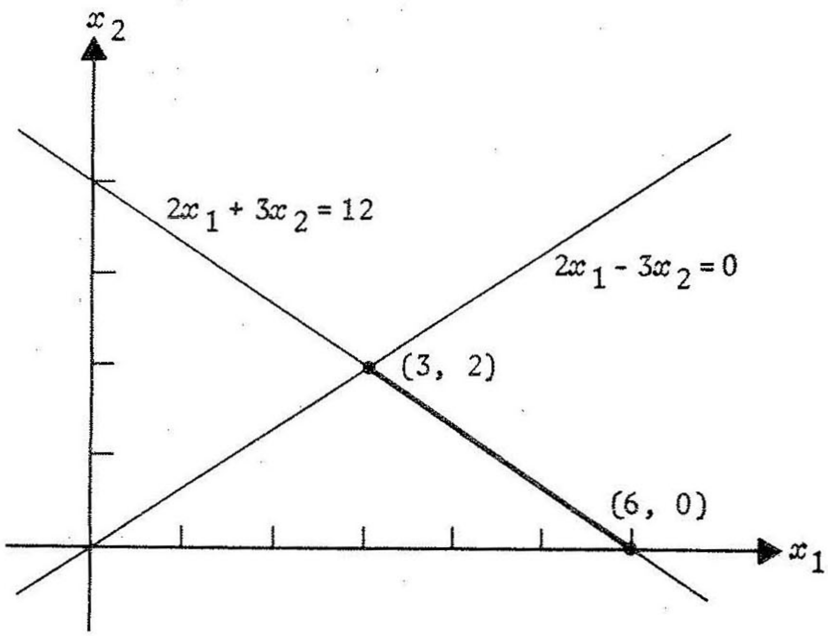

Figure 20.4

EXAMPLE 20.7 Find values of x_1 and x_2 which maximize

$$z = 2x_1 + 5x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 8 \\ -4x_1 + x_2 &\leq 2 \\ 2x_1 - 3x_2 &\leq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$

SOLUTION The feasible region of this linear programming problem is illustrated in Fig. 20.5. Since it is unbounded, we are not assured by Theorem 20.1 that the objective function attains a maximum value. In fact, it is easily seen that since the feasible region contains points for which both x_1 and x_2 are arbitrarily large and positive, then the objective function

$$z = 2x_1 + 5x_2$$

can be made arbitrarily large and positive. This problem has no optimal solution. Instead, we say the problem has an *unbounded solution*.

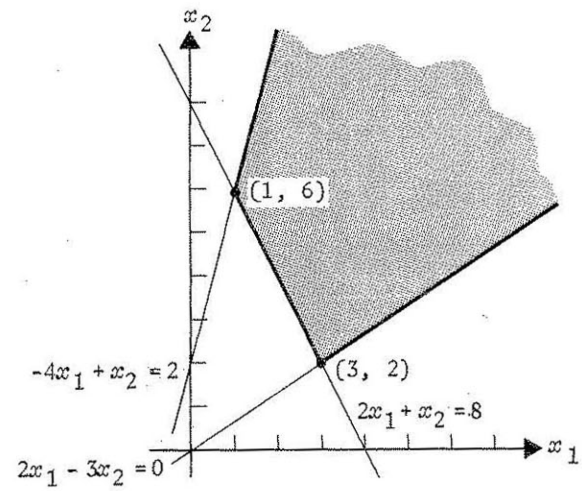

Figure 20.5

EXAMPLE 20.8 Find values of x_1 and x_2 which maximize

$$z = -5x_1 + x_2$$

subject to

$$2x_1 + x_2 \geq 8$$
$$-4x_1 + x_2 \leq 2$$
$$2x_1 - 3x_2 \leq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

SOLUTION The above constraints are the same as those in Example 20.7, so that the feasible region of this problem is also given by Fig. 20.5. In Exercise 20.10, we ask the reader to show that the objective function of this problem attains a maximum within the feasible region. By Theorem 20.1, this maximum must be attained at an extreme point. The values of z at the two extreme points of the feasible region are given by

Extreme point (x_1, x_2)	Value of $z = -5x_1 + x_2$
(1, 6)	1
(3, 2)	-13

The maximum value of z is thus 1 and is attained at the extreme point $x_1 = 1$, $x_2 = 6$.

EXAMPLE 20.9 Find the values x_1 and x_2 which minimize

$$z = 3x_1 - 8x_2$$

subject to

$$
\begin{aligned}
2x_1 - x_2 &\le 4 \\
3x_1 + 11x_2 &\le 33 \\
3x_1 + 4x_2 &\ge 24 \\
x_1 &\ge 0 \\
x_2 &\ge 0.
\end{aligned}
$$

SOLUTION As can be seen from Fig. 20.6, the intersection of the five half-planes defined by the five constraints is empty. This linear programming problem has no feasible solutions since the constraints are inconsistant.

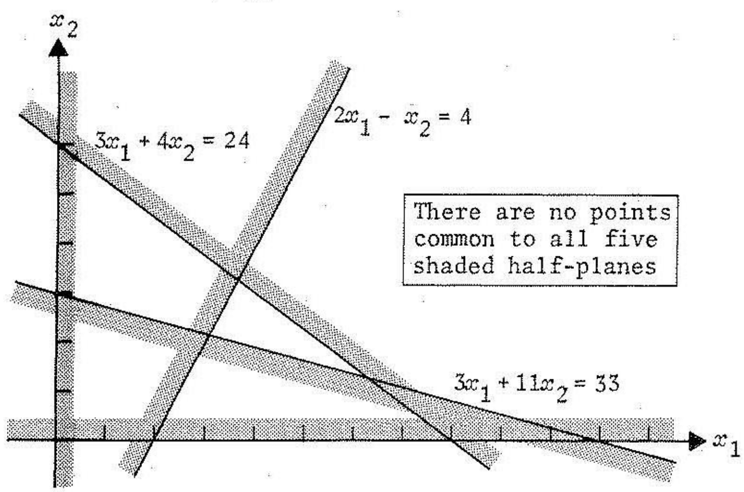

Figure 20.6

EXERCISES

20.1 Find values of x_1 and x_2 which maximize

$$z = 3x_1 + 2x_2$$

subject to

$$2x_1 + 3x_2 \leq 6$$
$$2x_1 - x_2 \geq 0$$
$$x_1 \leq 2$$
$$x_2 \leq 1$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

20.2 Find values of x_1 and x_2 which minimize

$$z = 3x_1 - 5x_2$$

subject to

$$2x_1 - x_2 \leq -2$$
$$4x_1 - x_2 \geq 0$$
$$x_2 \leq 3$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

20.3 Find values of x_1 and x_2 which minimize

$$z = -3x_1 + 2x_2$$

subject to

$$3x_1 - x_2 \geq -5$$
$$-x_1 + x_2 \geq 1$$
$$2x_1 + 4x_2 \geq 12$$
$$x_1 \geq 0$$
$$x_2 \geq 0.$$

20.4 Solve the linear programming problem posed in Example 20.2.

20.5 Solve the linear programming problem posed in Example 20.3.

20.6 A trucking firm ships the containers of two companies, A and B. Each container of Company A weighs 40 pounds and is 2 cubic feet in volume. Each container of Company B weighs 50 pounds and is 3 cubic feet in volume. The trucking firm charges Company A \$2.20 for each container shipped and charges Company B \$3.00 for each container shipped. If one of the firm's trucks cannot carry more than 37,000 pounds and cannot hold more than 2000 cubic feet, how many containers from companies A and B should a truck carry to maximize the shipping charges?

20.7 Repeat Exercise 20.6 if the trucking firm raises its price for shipping a container of Company A to \$2.50.

20.8 A manufacturer produces sacks of chicken feed from two ingredients, A and B. Each sack is to contain at least 10 ounces of nutrient N_1, at least 8 ounces of nutrient N_2, and at least 12 ounces of nutrient N_3. Each pound of ingredient A contains 2 ounces of nutrient N_1, 2 ounces of nutrient N_2, and 6 ounces of nutrient N_3. Each pound of ingredient B contains 5 ounces of nutrient N_1. 3 ounces of nutrient N_2, and 4 ounces of nutrient N_3. If ingredient A costs 8¢ per pound and ingredient B costs 9¢ per pound, how much of each ingredient should the manufacturer use in each sack of feed to minimize his costs?

20.9 If the objective function of a linear programming problem has the same value at two adjacent extreme points, show that it has the same value at all points on the straight-line segment connecting the two extreme points. Hint: if (x_1', x_2') and (x_1'', x_2'') are any two points in the plane, a point (x_1, x_2) lies on the straight-line segment connecting them if

$$x_1 = tx_1' + (1 - t)x_1''$$

and

$$x_2 = tx_2' + (1 - t)x_2''$$

where t is a number in the interval $[0, 1]$.

20.10 Show that the objective function in Example 20.8 attains a maximum value in the feasible set. Hint: Examine the level curves of the objective function.