Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Linear Algebra (Math 251)
Level IV, Assignment 3
(2016)

1. State whether the following statements are true or false: [6]

   (a) The sum of eigenvalues of a square matrix is same as its determinant.

   (a) False

   (b) (1,0,3) is the real part of the complex vector $(2i+1, -3i, i+3)$.

   (b) True

   (c) The inner product of a vector with itself can not be negative real number.

   (c) True

   (d) If u and v are orthogonal vector then the angle between these two vectors is zero.

   (d) False

   (e) If determinant of a matrix is 1 or -1, then the matrix is orthogonal.

   (e) False

   (f) In case of real matrices, Hermitian and symmetric matrices are same.

   (f) True

2. Select one of the alternatives from the following questions as your answer. [6]

   (a) If $u = (3, -1, 4),\ v = (0, 4, 6)$ and $k = 2$, then the value of $< ku, v >=$

   A. 20
   B. -20
   C. 40
   D. -40

(b) If $p = 4 + 3x - 2x^2$ be a vector in the vector space P_2, then $\|P\| =$

A. $\sqrt{7}$

B. $\sqrt{21}$

C. 5

D. $\sqrt{29}$

(c) The eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

A. $\{2, 3, 4\}$

B. $\{1, -1, 4\}$

C. $\{1, 0, -1\}$

D. $\{1, -1, 3\}$

(d) If $\{1, 4, -2\}$ be eigenvalues of a square matrix, then its determinant will be

A. -8

B. 3

C. 7

D. 8

(e) If $6x_1^2 + 3x_2^2 - 12x_1x_2$ be the quadratic form, then the associated symmetric matrix will be

A. $\begin{bmatrix} 6 & -6 \\ 3 & -6 \end{bmatrix}$

B. $\begin{bmatrix} 3 & -6 \\ -6 & 6 \end{bmatrix}$

C. $\begin{bmatrix} 6 & -12 \\ -12 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 6 & -6 \\ -6 & 3 \end{bmatrix}$

(f) For which value of a and b, the matrix $\begin{bmatrix} 3 & 1+i & 2+6i \\ a & -1 & 2-4i \\ 2-6i & b & 1 \end{bmatrix}$ is Hermitian?

A. $a = 1 + i, \quad b = 2 - 4i$

B. $a = 1 + i, \quad b = 2 + 4i$

C. $a = 1 - i, \quad b = 2 + 4i$

D. $a = 1 - i, \quad b = 2 - 4i$

3. Find all the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$. [4]

Solution: The characteristic equation of the matrix is given by

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \begin{bmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix} &= 0 \\ \lambda^2 - 3\lambda - 10 &= 0 \\ (\lambda - 5)(\lambda + 2) &= 0 \\ \lambda &= 5, \ -2 \end{aligned}$$

The roots 5 and -2 are eigenvalues of A.

Eigen Vectors

For $\lambda = 5$: Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 5$. So

$$\begin{aligned} (A - \lambda I)X &= 0 \\ \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ which\ gives \quad -x_1 + 2x_2 &= 0 \end{aligned}$$

This system has only one free variable x_2. Take $x_2 = 1$ so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is the required eigen vector corresponding to $\lambda = 5$.
Similarly, $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -2$.

4. If $u, v \in \mathbb{C}^2$ with $u = (1+i, 1-i), \ v = (3i, 3-i)$, then find $u.v$. [2]

Solution: Since u and v are complex vectors, so

$$\begin{aligned} u.v &= (1+i)\overline{3i} + (1-i)\overline{(3-i)} \\ &= -(1+i)3i + (1-i)(3+i) \\ &= -3i + 3 + 3 + i - 3i + 1 \\ &= 7 - 5i \end{aligned}$$

5. For $u = (1, 2, 3)$, $v = (4, 4, -4)$ and $w = (2, 1, 7)$, verify the property of inner product $< u + v, w > = < u, w > + < v, w >$. [3]

Solution: Since $u + v = (5, 6, -1)$ So

$$\begin{aligned} < u+v, w > &= 10 + 6 - 7 = 9 \\ < u, w > &= 2 + 2 + 21 = 25 \\ < v, w > &= 8 + 4 - 28 = -16 \\ < u, w > + < v, w > &= 25 - 16 = 9 \\ &= < u + v, w > \end{aligned}$$

Therefore, $< u + v, w > = < u, w > + < v, w >$.

6. Find the least squares solution of the system of linear equation $AX = B$, where $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. [3]

Solution: We know that for every linear system $AX = B$, the associated normal system $A^T AX = A^T B$ is consistent, and all its solutions least squares solutions of $AX = B$.

The associated normal system is given by

$$A^T AX = A^T B$$

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

which gives, $5x = 1$, $9y = -5$ and hence $x = 1/5$, $y = -5/9$ are the required least squares solution.

7. Show that the matrix $A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ is unitary. [3]

Solution: We know that a square complex matrix A is said to be unitary if $A^*A = I$, where A^* is the conjugate transpose of A. Given that

$$\begin{aligned}
A &= \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \\
So,\ A^* &= \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \\
Now,\ A^*A &= \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} (1+i)(1-i)+(1+i)(1-i) & (1+i)(1+i)+(1-i)(1-i) \\ (1+i)(1+i)+(1-i)(1-i) & (1+i)(1-i)+(1+i)(1-i) \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1-i^2+1-i^2 & 1+i^2+2i+1+i^2-2i \\ 1+i^2+2i+1+i^2-2i & 1-i^2+1-i^2 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 2+2 & 1-1+2i+1-1-2i \\ 1-1+2i+1-1-2i & 2+2 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

Hence A is an unitary matrix.

8. If $A = \begin{bmatrix} 1+i & 2i & -6 \\ i & 0 & 1-i \\ 2 & 7-6i & 13 \end{bmatrix}$, then find the conjugate transpose of B, where $B = iA$. [3]

Solution: Given that

$$A = \begin{bmatrix} 1+i & 2i & -6 \\ i & 0 & 1-i \\ 2 & 7-6i & 13 \end{bmatrix}$$

$$So, B = iA = i\begin{bmatrix} 1+i & 2i & -6 \\ i & 0 & 1-i \\ 2 & 7-6i & 13 \end{bmatrix}$$

$$= \begin{bmatrix} i+i^2 & 2i^2 & -6i \\ i^2 & 0 & i-i^2 \\ 2i & 7i-6i^2 & 13i \end{bmatrix}$$

$$= \begin{bmatrix} i-1 & -2 & -6i \\ -1 & 0 & i+1 \\ 2i & 7i+6 & 13i \end{bmatrix}$$

$$Now,\ \overline{B} = \begin{bmatrix} -i-1 & -2 & 6i \\ -1 & 0 & -i+1 \\ -2i & -7i+6 & -13i \end{bmatrix}$$

$$and\ hence\ \overline{B}^T = B^* = \begin{bmatrix} -1-i & -1 & -2i \\ -2 & 0 & 6-7i \\ 6i & 1-i & -13i \end{bmatrix}$$

which is the required matrix.