Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Linear Algebra (Math 251)
Level IV, Assignment 4
(Fall, 2016)

1. State whether the following statements are true or false: [6]

   (a) If $T : V \to W$ is a one to one linear transformation, then $ker(T) = \{0\}$.

   (a) True

   (b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map given by $T(x, y) = (x + y, y - 1)$, then T is linear.

   (b) False

   (c) Every square matrix can be decomposed into LU-decomposition.

   (c) False

   (d) If A is $m \times n$ matrix, then the eigen values of $A^T A$ can not be negative.

   (d) True

   (e) The following L.P.P has an unbounded feasible region.

$$\begin{aligned} \min\ z &= x - y \\ \textit{subject to}\quad 4x - 3y &\geq 0 \\ x + y &\leq 10 \\ x \geq 0,\ \ y &\geq 0. \end{aligned}$$

   (e) False

   (f) No L.P.P with an unbounded feasible region has a solution.

   (f) False

 Last Date: 31 Dec, 2016

2. Select one of the alternatives from the following questions as your answer. [6]

(a) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given by $T(x, y) = (2x - y, -4x + 2y)$, then which of the following vector is in $ker(T)$?

A. $(1, 4)$

B. $(2, 1)$

C. $(1, 1/2)$

D. $(1/2, 1)$

(b) If $T : W \to V$ be a linear transformation, then $ker(T)$ and $range(T)$ are subspaces of vector space(s)

A. V.

B. W.

C. W and V respectively.

D. V and W respectively.

(c) Which of the following sets of eigenvalues have a dominant eigenvalue:

A. $\{8, -7, -6, 8\}$

B. $\{-5, -2, 2, 4\}$

C. $\{-3, -2, -1, , 0, 1, 2, 3\}$

D. None of the above

(d) If $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ be a matrix where $B = A^T A$, then the singular values of A are

A. $\{4,\ 9,\ 0\}$

B. $\{0,\ 9,\ 16\}$

C. $\{4,9,16\}$

D. $\{2, 3, 4\}$

(e) In linear programming, objective function and objective constraints are

A. solved.

B. quadratic.

C. adjacent.

D. linear.

(f) The feasible region

A. is defined by the objective function.

B. is an area bounded by the collective constraints and represents all permissible combinations of the decision variables.

C. represents all values of each constraint.

D. may range over all positive or negative values of only one decision variable.

3. Consider the basis $S = \{v_1, v_2\}$ for $\mathbb{R}^2$, where $v_1 = \{1, 1\}$ and $v_2 = \{1, 0\}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator for which $T(v_1) = (1, 2)$, and $T(v_2) = (3, 0)$. Find a formula for $T(x_1, x_2)$ and use it to find $T(2, -4)$. [4]

Solution: Take any element $(x_1, x_2) \in \mathbb{R}^2$, so we can write:

$$\begin{aligned}
(x_1, x_2) &= c_1v_1 + c_2v_2 \\
&= c_1(1, 1) + c_2(1, 0) = (c_1 + c_2, c_1) \\
On\ simplyfying,\ \ we\ get & \\
c_1 = x_2, &\ \ c_2 = x_1 - x_2 \\
Again,\ \ (x_1, x_2) &= c_1v_1 + c_2v_2 \\
Since\ T\ is\ linear,\ so & \\
T(x_1, x_2) &= c_1T(v_1) + c_2T(v_2) \\
&= c_1(1, 2) + c_2(3, 0) \\
&= (c_1, 2c_1) + (3c_2, 0) \\
&= (c_1 + 3c_2, 2c_1) \\
On\ simplyfying,\ \ we\ get & \\
&= (3x_1 - 2x_2, 2x_2) \\
T(x_1, x_2) &= (3x_1 - 2x_2, 2x_2).
\end{aligned}$$

Which is the required formula.

Now, $T(2, -4) = (6 + 8, -8) = (14, -8)$.

4. Check whether the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (xy, x)$ is linear or not. [3]

Solution: Let $v_1 = (x_1, y_1),\ \ v_2 = (x_2, y_2) \in \mathbb{R}^2$, so $v_1 + v_2 = (x_1 + x_2, y_1 + y_2)$.

$$\begin{aligned}
T(v_1) &= T(x_1, y_1) = (x_1y_1, x_1) \\
T(v_2) &= T(x_2, y_2) = (x_2y_2, x_2) \\
T(v_1) + T(v_2) &= (x_1y_1 + x_2y_2, x_1 + x_2) \\
T(v_1 + v_2) &= T(x_1 + x_2, y_1 + y_2) = ((x_1 + x_2)(y_1 + y_2),\ x_1 + x_2) \\
&= (x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1,\ x_1 + x_2) \\
&\neq T(v_1) + T(v_2) \\
T(v_1 + v_2) &\neq T(v_1) + T(v_2)
\end{aligned}$$

Therefore T is not linear.

5. Find an LU-decomposition of matrix $A = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}$. [4]

Solution: We know that "Every square matrix can be decomposed into LU form where L is a unit lower triangular matrix and U is the upper triangular matrix provided that all the principal minors are non zero."
Since

$$A = LU$$
$$\begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ 0 & u_3 \end{bmatrix}$$
$$= \begin{bmatrix} u_1 & u_2 \\ l_1 u_1 & l_1 u_2 + u_3 \end{bmatrix}$$

Two matrices are equal only when the corresponding elements are equal. So, we have
$u_1 = 5, \quad u_2 = -1, \quad l_1 u_1 = -1, \quad l_1 u_2 + u_3 = -1$
Therefore we have
$l_1 = -\frac{1}{5}, \quad u_3 = -\frac{6}{5}$.
Hence, $L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{5} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 5 & -1 \\ 0 & -\frac{6}{5} \end{bmatrix}$.

which is the required LU decomposition.

6. Find the singular values of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. [3]

Solution: The first step is to find the eigenvalues of the matrix

$$A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

The characteristic polynomial of $A^T A$ is

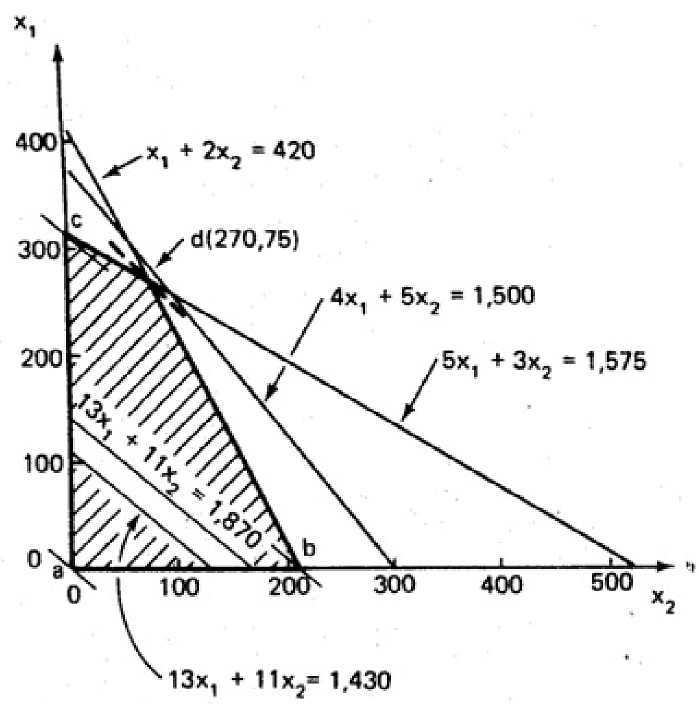
$$\lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1)$$

so the eigenvalues of $A^T A$ are 9 and 1 and the singular values of A in order of decreasing size are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$.

7. Solve the following LPP by graphical method: [4]

$$\max z = 13x_1 + 11x_2 \quad \text{subject to :}$$
$$4x_1 + 5x_2 \leq 1500$$
$$5x_1 + 3x_2 \leq 1575$$
$$x_1 + 2x_2 \leq 420$$
$$x_1,\ x_2 \geq 0.$$

Solution: Graph the constraints as lines and find the feasible region.



At the corner point $d(270, 75)$ we get the maximum value of $z = 4335$.

 End of Assignment.