Ministry of Higher Education
Kingdom of Saudi Arabia

CSTS
SEU, KSA

Linear Algebra (Math 251)
Level IV, Assignment 3
(Fall, 2016)

1. State whether the following statements are true or false: [6]

   (a) If 2, 3 and 4 are eigen values of a matrix A, then $\det(A) = 9$.

   (a) False

   (b) (1,-1,2) is the real part of the complex vector $(1+i, -1+i, i, 2)$.

   (b) False

   (c) The inner product of a nonzero vector with itself is always a positive real number.

   (c) True

   (d) If $u = (1, -1)$, $v = (-2, 2)$ and $k = 4$, then $< ku, v >=$ 16.

   (d) False

   (e) If determinant of a matrix is 1 or -1, then the matrix is orthogonal.

   (e) False

   (f) The rows and columns of an orthogonal matrix are orthonormal.

   (f) True

 Last Date: 14 Dec, 2016

2. Select one of the alternatives from the following questions as your answer. [6]

   (a) The characteristic equation of the matrix $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ is

      A. $\lambda^2 - 7\lambda - 10 = 0$
      B. $\lambda^2 + 7\lambda - 10 = 0$
      C. $\lambda^2 - 7\lambda + 10 = 0$
      D. $\lambda^2 + 7\lambda + 10 = 0$

   (b) The eigenvalues of the matrix A^3, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & -3 \end{bmatrix}$, are

      A. $\{1, 4, -3\}$
      B. $\{1, 12, -9\}$
      C. $\{1, 64, 27\}$
      D. $\{1, 64, -27\}$

   (c) Which of the following sets of vectors are orthogonal with respect to the Euclidean inner product on $\mathbb{R}^2$:

      A. (1,2), (-2,1)
      B. (3,4),(2,6)
      C. (6,9),(5,2)
      D. (0,4), (0,6)

   (d) If angle between vectors u and v is zero such that $\|u\| = 4,\ \|v\| = 6$, then $< u, v >=$

      A. 10
      B. 24
      C. $\sqrt{24}$
      D. $\sqrt{10}$

   (e) If $3x_1^2 + 2x_2^2 - 4x_3^2 - 2x_1x_2 + 6x_1x_3 - 4x_2x_3$ be the quadratic form, then the associated symmetric matrix will be

      A. $\begin{bmatrix} 3 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & -4 \end{bmatrix}$
      B. $\begin{bmatrix} 3 & -1 & -3 \\ -1 & 2 & -2 \\ -3 & -2 & -4 \end{bmatrix}$
      C. $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 2 & 2 \\ 3 & 2 & -4 \end{bmatrix}$

D. $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & -4 \end{bmatrix}$

(f) For which value of a and b, the matrix $\begin{bmatrix} 3 & i+2 & 2+6i \\ a & -1 & 2i-1 \\ 2-6i & b & 1 \end{bmatrix}$ is Hermitian?

A. $a = i - 2, \quad b = -2i - 1$
B. $a = -i + 2, \quad b = 2i + 1$
C. $a = -i + 2, \quad b = -2i - 1$
D. $a = i - 2, \quad b = 2i + 1$

3. Find all the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$. [4]

Solution: The characteristic equation of the matrix is given by

$$\det(A - \lambda I) = 0$$
$$\begin{bmatrix} 5-\lambda & -2 \\ 3 & 0-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 2,\ 3$$

The roots 2 and 3 are eigenvalues of A.

Eigen Vectors

For $\lambda = 2$: Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to $\lambda = 2$. So

$$(A - \lambda I)X = 0$$
$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\textit{which gives} \quad 3x_1 - 2x_2 = 0$$

This system has only one free variable x_2. Take $x_2 = 1$, which gives $x_1 = 2/3$ and so $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ is the required eigen vector corresponding to $\lambda = 2$.

Similarly, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$.

4. For the matrix $A = \begin{bmatrix} 1+3i & 2 \\ 3+i & 4-i \end{bmatrix}$ Find $\overline{A}$, $Re(A)$, $Im(A)$, $Tr(A)$ and $\det(A)$. [2]

Solution: $\overline{A} = \begin{bmatrix} 1-3i & 2 \\ 3-i & 4+i \end{bmatrix}$

$Re(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $Im(A) = \begin{bmatrix} 3 & 0 \\ 1 & 4-1 \end{bmatrix}$

$Tr(A) = 5 + 2i$.

$\det(A) = (1+3i)(4-i) - 2(3-i) = 1 + 9i$.

5. Find the value of k, for which the matrices $U = \begin{bmatrix} 9 & 4 \\ 2 & 6 \end{bmatrix}$ and $V = \begin{bmatrix} 2 & -2 \\ k & 4 \end{bmatrix}$ are orthogonal in the vector space $M_{2\times 2}$ with usual inner product on $M_{2\times 2}$. [2]

Solution: Since U, V are orthogonal, so

$$\begin{aligned} < U, V > &= 0 \\ 18 - 8 + 2k + 24 &= 0 \\ 2k &= -34 \\ k &= -\,17. \end{aligned}$$

6. Find the least squares solution of the system of linear equation $AX = B$, where $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. [4]

Solution: We know that for every linear system $AX = B$, the associated normal system $A^T AX = A^T B$ is consistent, and all its solutions are least squares solutions of $AX = B$.

The associated normal system is given by

$$A^T AX = A^T B$$

$$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

which gives, $9x = 3$, $5y = 8$ and hence $x = 1/3$, $y = 8/5$ are the required least squares solution.

7. Show that the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ is unitary. [3]

Solution: We know that a square complex matrix A is said to be unitary if $A^*A = I$, where A^* is the conjugate transpose of A. Given that

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}. \qquad \text{So,} \quad A^* = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\text{Now, } A^*A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ -\frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{2}{3} & \frac{1-i}{3} - \frac{1-i}{3} \\ \frac{1+i}{3} - \frac{1+i}{3} & \frac{2}{3} + \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence A is an unitary matrix.

8. Show that matrix $A = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$ is orthogonal and find A^{-1}. [3]

Solution: Given that $A = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$, so $A^T = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$.

$$\text{Now, } AA^T = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence A is orthogonal and $A^{-1} = A^T$.

 End of Assignment.