SEC (5.1) EIGENVALUES AND EIGENVECTORS.

In this Section, we will define the notions of 'Eigenvalue' and 'Eigenvector' and discuss some of their basic properties.

Definition of Eigenvalue and Eigenvector

If 'A' is an $n \times n$ matrix, then a non-zero vector $\mathbf{x}$ in R^n is called an Eigenvector of 'A' (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$, that is,

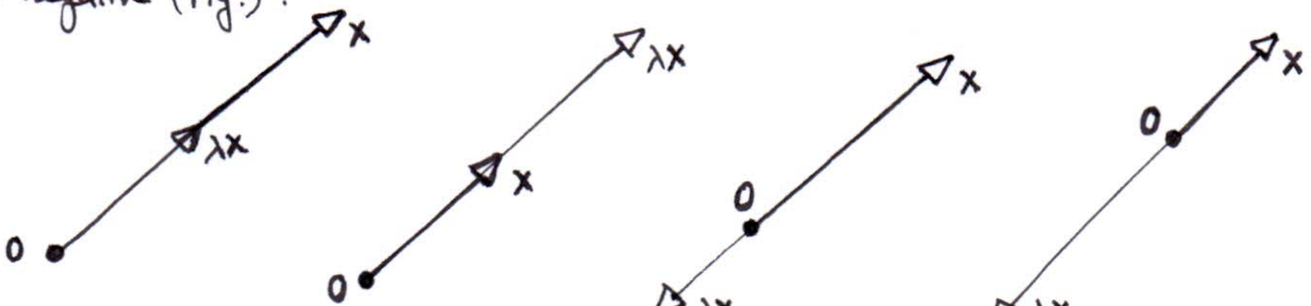
$$A\mathbf{x} = \lambda \mathbf{x} \quad , \text{ for some scalar } \lambda .$$

The scalar λ is called an Eigenvalue of 'A' (or of T_A),

and $\mathbf{x}$ is said to be an Eigenvector corresponding to λ.

NOTE: In general, the image of a vector $\mathbf{x}$ under multiplication by a square matrix 'A' differs from $\mathbf{x}$ in both magnitude and direction. However, in the special case where $\mathbf{x}$ is an eigenvector of 'A', multiplication by 'A' leaves the direction unchanged.

For example, in R^2 or R^3, multiplication by 'A' maps each eigenvector $\mathbf{x}$ of A (if any) along the same line through the origin as $\mathbf{x}$. Depending on the sign and magnitude of the eigenvalue λ corresponding to $\mathbf{x}$, the operation $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches $\mathbf{x}$ by a factor of λ, with a reversal of direction in the case where λ is negative (fig.).

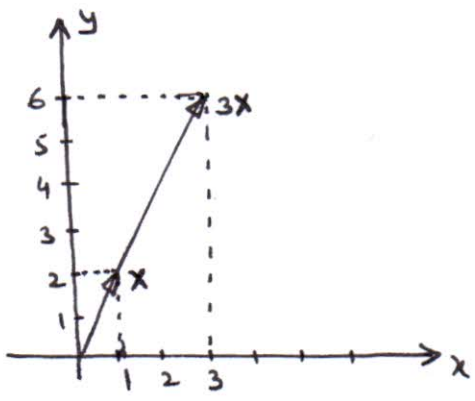


(i) $0 \le \lambda \le 1$ (ii) $\lambda \geq 1$ (iii) $-1 \le \lambda \le 0$ (iv) $\lambda \le -1$

Example ① Eigenvector of a 2×2 Matrix

The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an Eigenvector of matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ corresp. to eigenvalue $\lambda = 3$

since
$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= 3\mathbf{x} \end{aligned}$$



Geometrically, multiplication by A has stretched the vector $\mathbf{x}$ by a factor of 3 (See fig.)

COMPUTING EIGENVALUES AND EIGENVECTORS

Our next objective is to obtain a general procedure for finding Eigenvalues and Eigenvectors of an $n \times n$ matrix 'A'. We will begin with the problem of finding the Eigenvalues of 'A'.

Note first that the equ. $A\mathbf{x} = \lambda \mathbf{x}$ can be rewritten as $A\mathbf{x} = \lambda I \mathbf{x}$

or equivalently as $(\lambda I - A)\mathbf{x} = 0$

For λ to be an Eigenvalue of A, this equ. must have a non-zero solu. for $\mathbf{x}$. But it follows from a Theorem of Chapter ④ that this is so iff the coefficient matrix $(\lambda I - A)$ has a zero determinant. Thus, we have the following result —

THEOREM① If 'A' is an $n \times n$ matrix, then λ is an Eigenvalue of A iff it satisfies the equ.

$$\det(\lambda I - A) = 0 \qquad \text{———— ①}$$

This equ. is called Characteristic Equation of matrix A.

Example ② Use Characteristic Equ. to find all eigenvalues of matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

Solu. The eigenvalues of the matrix A are the solutions of the equ.

$$|\lambda I - A| = 0$$

i.e., $$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = 0$$

i.e., $$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = 0$$

i.e., $$\begin{vmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda-3)(\lambda+1) = 0 \quad \Rightarrow \quad \underline{\lambda = 3, -1}$$

NOTE: When $\det(\lambda I - A)$ on left side of ① is expanded, the result is a polynomial $p(\lambda)$ of degree n, that is called Characteristic Polynomial of matrix A. For example, it follows that the characteristic polynomial of 2×2 matrix A in Example ② is

$$p(\lambda) = (\lambda-3)(\lambda+1)$$
$$= \lambda^2 - 2\lambda - 3, \text{ which is a polynomial of degree 2.}$$

In general, the Characteristic Polynomial of an $n \times n$ matrix has the form

$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$, in which the coeff. of λ^n is 1.

Since a polynomial of degree n has at most n distinct roots, it follows that the equ.

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0 \qquad \text{———— ②}$$

has at most n distinct solutions and consequently that an $n \times n$ matrix has at most n distinct eigenvalues. Since some of these solu. may be complex numbers, it is possible for a matrix to have complex eigenvalues, even if that matrix itself has real entries.

Example ③ Eigenvalues of a 3x3 Matrix

Find the Eigenvalues of the matrix, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.

Soln. The Characteristic Polynomial of matrix A is

$$\det(\lambda I - A)$$

$$= \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} \right)$$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{vmatrix}$$

$$= \lambda[\lambda(\lambda-8)+17] - (-1)[0-(-4)(-1)]$$

$$= \lambda(\lambda^2 - 8\lambda + 17) + (0-4)$$

$$= \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The Eigenvalues of A must therefore satisfy the cubic eqn.

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \quad \text{——— ①}$$

putting $\lambda = 4$ in L.H.S. of ①, we get

We try only divisors of cont. term (i.e. -4) which are $\pm1, \pm2, \pm4$

$$\text{L.H.S.} = (4)^3 - 8(4)^2 + 17(4) - 4$$

$$= 64 - 128 + 68 - 4 = 0 = \text{R.H.S.}$$

so $\lambda - 4$ must be a factor of L.H.S. of ①

Now dividing $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ by $\lambda - 4$ as:-

$$\begin{array}{r|l} & \lambda^2 - 4\lambda + 1 \\ \hline \lambda - 4 & \lambda^3 - 8\lambda^2 + 17\lambda - 4 \\ & \underline{\lambda^3 - 4\lambda^2} \\ & \quad -\ \ + \\ & -4\lambda^2 + 17\lambda - 4 \\ & \underline{-4\lambda^2 + 16\lambda} \\ & \quad +\ \ - \\ & \lambda - 4 \\ & \underline{\lambda - 4} \\ & \quad -\ \ + \\ & 0 \end{array}$$

Hence ① can be rewritten as

$$(\lambda-4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\Rightarrow \lambda = 4 \quad \text{and} \quad \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4}}{2}$$

i.e., $\lambda = 4$ and $\lambda = \dfrac{4 \pm \sqrt{12}}{2}$

i.e., $\lambda = 4$ and $\lambda = \dfrac{4 \pm 2\sqrt{3}}{2}$

i.e., $\lambda = 4$ and $\lambda = 2 \pm \sqrt{3}$

Thus, Eigenvalues of matrix A are $\lambda_1 = 4$, $\lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2 - \sqrt{3}$.

Example ④ Eigenvalues of an Upper Triangular Matrix

Find the Eigenvalues of the upper triangular matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$

Soln. Recalling that the determinant of a triangular matrix is equal to the product of the entries on the main diagonal, we obtain

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \right)$$

$$= \begin{vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{vmatrix}$$

$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44})$$

Thus, the Characteristic Eqn. is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

$$\Rightarrow \lambda = a_{11}, \ \lambda = a_{22}, \ \lambda = a_{33}, \ \lambda = a_{44}$$

Thus, the Eigenvalues are $\lambda_1 = a_{11}, \ \lambda_2 = a_{22}, \ \lambda_3 = a_{33}, \ \lambda_4 = a_{44}$

which are precisely the diagonal entries of matrix A.

The following general theorem should be evident from the computations in preceding example–

THEOREM ②: If 'A' is $n \times n$ triangular matrix (upper triangular, lower triangular or diagonal), then the Eigenvalues of matrix A are the entries on the main diagonal of 'A'.

Example ⑤ Eigenvalues of a Lower Triangular Matrix

Find the eigenvalues of lower triangular matrix $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -1/4 \end{bmatrix}$

Soln. By Theorem ②, the eigenvalues of matrix A are

$$\lambda_1 = \frac{1}{2}, \ \lambda_2 = \frac{2}{3}, \ \lambda_3 = -\frac{1}{4}.$$

THEOREM ③ If A is an $n \times n$ matrix, the following statements are equivalent –

(i) 'λ' is an eigenvalue of matrix A.

(ii) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.

(iii) There is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda \mathbf{x}$.

(iv) 'λ' is a solution of the characteristic eqn. $\det(\lambda I - A) = 0$.

FINDING EIGENVECTORS AND BASES FOR EIGENSPACES.

Since the eigenvectors corresponding to an eigenvalue λ of a matrix 'A' are the non-zero vectors that satisfy the equ. $(\lambda I - A)x = 0$

These Eigenvectors are the non-zero vectors in the null space of matrix $\lambda I - A$. We call this null space the Eigenspace of 'A' corresponding to λ.

Stated another way, the Eigenspace of a matrix A corresponding to eigenvalue λ is the solution space of the homogeneous system $(\lambda I - A)x = 0$.

<u>NOTE</u>: Notice that $x = 0$ is in every eigenspace even though it is not an eigenvector. Thus, it is the non-zero vectors in an eigenspace that are the eigenvectors.

Example ⑥ Bases for Eigenspaces.

Find bases for the eigenspaces of the matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$.

<u>Solu.</u> The Characteristic Equ. of A is

$$(\lambda - 3)(\lambda + 1) = 0 \quad , \quad \text{Do as Example ②}$$

$$\Rightarrow \lambda = 3, -1 \quad (\text{Two distinct eigenvalues})$$

Thus, there are two eigenspaces of 'A', one corresponding to each of these eigenvalues.

~~Basis for~~ The eigenvectors corresponding to eigenvalue $\lambda = 3$ is the non-trivial solu. x of the equ.

$$(\lambda I - A)x = 0$$

ie., $$(3I - A)x = 0$$

$$\Rightarrow \left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -8x_1 + 4x_2 = 0 \quad \text{—— ①}$$

Let $x_2 = t$, then from ① $\quad -8x_1 + 4t = 0 \Rightarrow x_1 = \frac{1}{2}t$

Thus, the ~~basis for the eigenspace~~ Eigenvector corresponding to $\lambda = 3$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix}$$

ie., $$x = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$\therefore$ the basis for the eigenspace corresponding to $\lambda = 3$ is $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$.

Similarly, we can find a basis for eigenspace corresponding to $\lambda = -1$ given as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example ⑦ Eigenvectors and Bases for Eigenspaces

Find bases for eigenspaces of $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Soln. The Characteristic Equ. for matrix A is

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right) = 0$$

$$\Rightarrow \quad \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda-2)\begin{vmatrix} \lambda & 2 \\ -1 & \lambda-3 \end{vmatrix} = 0 \quad , \text{ expanding det. by } C_2$$

$$\Rightarrow \quad (\lambda-2)[\lambda(\lambda-3) - 2(-1)] = 0$$

$$\Rightarrow \quad (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0$$

$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow \quad (\lambda-1)(\lambda^2 - 4\lambda + 4) = 0 \quad , \text{ by factorization as before}$$

$$\Rightarrow \quad (\lambda-1)(\lambda-2)^2 = 0 \quad \Rightarrow \underline{\lambda = 1, 2, 2}$$

Thus, the distinct eigenvalues of A are $\lambda = 1$ & $\lambda = 2$, so there are two eigenspaces of A.

By defi., $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A iff $\mathbf{x}$ is a non-trivial solution of

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— ①}$$

In the case where $\lambda = 2$, formula ① becomes

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad , R_1 \rightarrow R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad , R_2 \to R_2 + R_1 \quad R_3 \to R_3 + R_1$$

$$\Rightarrow \quad x_1 + 0x_2 + x_3 = 0 \quad \text{———} \quad ②$$

Let $x_2 = t$, $x_3 = s$ then from ② $x_1 = -s$

∴ The soln. of system of eqn. ① is $x_1 = -s$, $x_2 = t$, $x_3 = s$

Thus, the Eigenvectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form

$$X = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s + 0t \\ 0s + t \\ s + 0t \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent, these vectors form a basis for the eigenspace corresponding to the eigenvalue $\lambda = 2$.

<u>If $\lambda = 1$, then ① becomes</u>

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad , R_2 \to R_2 + R_1 \quad , R_3 \to R_3 + R_1$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad , R_2 \to (-1) R_2$$

$$\therefore \quad x_1 + 2x_3 = 0$$
$$x_2 - x_3 = 0$$

Writing these equations for leading variables in terms of free variables as

$$x_1 = -2x_3 \quad \text{———} \quad (i)$$
$$\& \quad x_2 = x_3 \quad \text{———} \quad (ii)$$

Let $x_3 = s$, then from (ii) $x_2 = s$

& from (i) $x_1 = -2s$

Thus, the Eigenvectors corresponding to $\lambda = 1$ are the non-zero vectors of the form

$$X = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Thus $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$.

POWERS OF A MATRIX : Once the eigenvalues and eigenvectors of a matrix A are found, it is a simple matter to find the eigenvalues and eigenvectors of any positive integer power of A;

for example, if λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

which shows that λ^2 is an eigenvalue of A^2 and that $\mathbf{x}$ is a corresponding eigenvector.

In general, we have the following result —

THEOREM 4 If k is a positive integer, λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresp. eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresp. eigenvector.

Example 8 Powers of a Matrix

In Example 7, we showed that the eigenvalues of the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ are $\lambda = 2$ and $\lambda = 1$.

so from Theorem 4, both $\lambda = 2^7 = 128$ and $\lambda = 1^7 = 1$ are eigenvalues of A^7.

We also showed that $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are eigenvectors of A corresp. to $\lambda = 2$

so from Theorem 4, they are also eigenvectors of A^7 corresp. to $\lambda = 2^7 = 128$.

Similarly, the eigenvector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ of A corresp. to the eigenvalue $\lambda = 1$ is also an eigenvector of A^7 corresponding to $\lambda = 1^7 = 1$.

EIGENVALUES AND INVERTIBILITY : The next theorem establishes a relationship between eigenvalues and the invertibility of a matrix.

THEOREM 5 A square matrix A is Invertible iff $\lambda = 0$ is not an eigenvalue of A.

Example 9 Eigenvalues and Invertibility

The matrix A in Example 7 is Invertible since it has eigenvalues $\lambda = 1$ & $\lambda = 2$, neither of which is zero. We can check this conclusion by showing that $\det(A) \neq 0$.

SEC 5.2 DIAGONALIZATION

In this Section, we will be concerned with the problem of finding a basis for R^n that consists of eigenvectors of an $n\times n$ matrix A. Such bases can be used to study geometric properties of A and to simplify various numerical computations.

THE MATRIX DIAGONALIZATION PROBLEM: Our first objective in this Section is to show that the following two seemingly different problems are equivalent.

Problem ① Given an $n\times n$ matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

Problem ② Given an $n\times n$ matrix A, does A have n linearly independent eigenvectors?

SIMILARITY: The matrix product $P^{-1}AP$ is called a Similarity Transformation of the matrix A. Such products are important in the study of eigenvectors & eigenvalues, so we will begin with some terminology about them.

Definition: If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$.

NOTE: Note that if B is similar to A, then it is also true that A is similar to B. We will usually say that A & B are Similar Matrices if either is similar to the other.

SIMILARITY INVARIANTS. Similar matrices have many properties in common. For example, if $B = P^{-1}AP$, then it follows that A & B have the same determinant, since

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det(A).$$

In general, any property that is shared by all similar matrices is called a Similarity Invariant or is said to be Invariant under Similarity.

TABLE ① Similarity Invariants

Property	
i) Determinant	A and $P^{-1}AP$ have same determinant.
ii) Invertibility	A is Invertible iff $P^{-1}AP$ is Invertible.
iii) Rank	A and $P^{-1}AP$ have same Rank.
iv) Nullity	A and $P^{-1}AP$ have same Nullity.
v) Trace	A and $P^{-1}AP$ have same Trace.
vi) Characteristic Polynomial	A and $P^{-1}AP$ have same Characteristic Polynomial.
vii) Eigenvalues	A and $P^{-1}AP$ have same eigenvalues.
viii) Eigenspace dimension	If λ is an eigenvalue of A and hence of $P^{-1}AP$, then eigenspace of A corresp. to λ and eigenspace of $P^{-1}AP$ corresp. to λ have same dimension.

DIAGONALIZABLE: A square matrix 'A' is said to be Diagonalizable if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case, the matrix P is said to Diagonalize 'A'.

The following theorem shows that Problems ① & ② posed before are actually two different forms of the same mathematical problem.

THEOREM ① If 'A' is an $n\times n$ matrix, the following statements are equivalent –
(i) 'A' is diagonalizable.
(ii) 'A' has n linearly independent eigenvectors.

NOTE: Part (ii) of Theorem ① is equivalent to saying that there is a Basis for R^n consisting of eigenvectors of A.

PROCEDURE FOR DIAGONALIZING A MATRIX

Step ① Confirm that the matrix is actually diagonalizable by finding n linearly independent eigenvectors. One way to do this is by finding a basis for each eigenspace and merging these basis vectors into a single set S. If this set has fewer than n vectors, then the matrix is not diagonalizable.

Step ② Form the matrix $P = [p_1 \; p_2 \; \cdots \; p_n]$ that has vectors in S as its column vectors.

Step ③ The matrix $P^{-1}AP$ will be diagonal and have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to the eigenvectors $p_1, p_2, \ldots, p_n$ as its successive diagonal entries.

Example ① Finding a Matrix P that Diagonalizes a Matrix 'A'

Find a matrix P that diagonalizes the matrix $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Soln. In Example ⑦ of Section (5.1), we found the eigenvalues of A are $\lambda = 1, 2, 2$ and we found the following bases for the eigenspaces: –

$$\lambda = 2: \quad p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1: \quad p_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalizes A.
As a check, we can verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{Eigenvalues of } A \text{ are on the diagonal.}$$

Example ② A Matrix That is not Diagonalizable

Find a matrix P that diagonalizes the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$.

Soln. The characteristic Equ. of matrix A is

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda-1)(\lambda-2)(\lambda-2) = 0$$

Thus, the distinct eigenvalues of matrix A are $\lambda=1$ and $\lambda=2$.

By Defn., $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A iff $\mathbf{x}$ is a non-trivial solution of the equ.

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

i.e.,

$$\begin{bmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{— ①}$$

When $\lambda=1$, Equ. ① becomes

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_1 \to (-1)R_1 \\ R_2 \leftrightarrow R_3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad , \quad R_2 \to \left(-\tfrac{1}{8}\right) R_2$$

$\therefore$ Corresponding equations are

$$x_1 + x_2 = 0 \quad \text{——— (i)}$$
$$x_2 + \tfrac{1}{8} x_3 = 0 \quad \text{——— (ii)}$$

Let $x_3 = s$, then from (ii) , $x_2 = -\frac{1}{8} s$

& from (i) , $x_1 = \frac{1}{8} s$

$\therefore$ Eigenvector corresponding to $\lambda = 1$ is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} s \\ -\frac{1}{8} s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$$

Thus, $\begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 1$

Similarly, we can show that

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 2$.

Since the matrix A is a 3×3 matrix and there are only two basis vectors in total, so matrix A is not diagonalizable.

THEOREM ② If $v_1, v_2, \cdots, v_k$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, then $\{v_1, v_2, \cdots, v_k\}$ is a linearly independent set.

THEOREM ③ If an $n \times n$ matrix A has n distinct eigenvalues, then A is Diagonalizable.

Example ⑤ Diagonalizability of Triangular Matrices

From a Theorem of Sec 5.1, we know that the eigenvalues of a triangular matrix are the entries on its main diagonal. Thus, a triangular matrix with distinct entries on the main diagonal is Diagonalizable.

For example, the matrix $A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = -2$ (four distinct eigenvalues)

The matrix A is 4x4 matrix containing 4 distinct eigenvalues.

so A is Diagonalizable.

COMPUTING POWERS OF A MATRIX: There are many applications in which it is necessary to compute high powers of a square matrix A. We will show next that if A happens to be diagonalizable, then the computations can be simplified by diagonalizing A.

To start, suppose that A is a diagonalizable $n \times n$ matrix and P diagonalizes A

then
$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

Squaring both sides,
$$(P^{-1}AP)^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{bmatrix} = D^2$$

$$\boxed{\begin{aligned} &(P^{-1}AP)^2 \\ &= P^{-1}AP\,P^{-1}AP \\ &= P^{-1}AIAP \\ &= P^{-1}A^2P \end{aligned}}$$

$$P^{-1}A^2P = D^2$$

More generally, if k is a positive integer, then a similar computation show that

$$P^{-1}A^kP = D^k$$

which we can rewrite as
$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} P^{-1} \quad \text{—①}$$

Formula ① reveals that raising a diagonalizable matrix A to a positive integer power has the effect of raising its eigenvalues to that power.

Example ④ Use ① to find A^{13} where $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

Soln. We showed in Example ① that the matrix A is diagonalized by the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and that
$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus it follows that

$$A^{13} = P D^{13} P^{-1}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8191 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

SEC (5.3) COMPLEX VECTOR SPACES

Because the characteristic equ. of any square matrix can have complex solutions, the notions of complex eigenvalues and eigenvectors arise naturally, even within the context of matrices with real entries. In this Section, we will discuss this idea and apply our results to study symmetric matrices in more detail.

Review of Complex No.'s

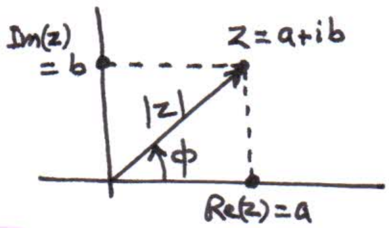
Recall that if $z = a + ib$ is a complex no., then

* $Re(z) = a$ and $Im(z) = b$ are called Real Part of z and Imaginary part of z, resp.
* $|z| = \sqrt{a^2+b^2}$ is called the Modulus (or absolute value) of z.
* $\bar{z} = a - ib$ is called the Complex Conjugate of z.

and
$$z\bar{z} = (a+ib)(a-ib) = a^2+b^2 = |z|^2$$

* The angle ϕ in Fig. is called an Argument of z,

$$Re(z) = |z|\cos\phi$$
$$Im(z) = |z|\sin\phi$$



* $z = |z|(\cos\phi + i\sin\phi)$ is called the Polar form of z.

COMPLEX EIGENVALUES.

We observed (from Sec (5.1)) that the characteristic equ. of a general $n\times n$ matrix A has the form
$$\lambda^n + c_1\lambda^{n-1} + \cdots\cdots + c_n = 0 \quad \text{——— ①}$$
in which the highest power of λ has a coefficient of 1.

Upto now, we have limited our discussion to matrices in which the solutions of ① are real numbers. However, it is possible for the characteristic equ. of a matrix A with real entries to have imaginary solutions;

for example, the characteristic equ. of the matrix $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$

is $\det(\lambda I - A) = 0$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} \lambda+2 & 1 \\ -5 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda-2) - (-5)\times 1 = 0$$

$$\Rightarrow \quad \lambda^2 + 1 = 0$$

$$\Rightarrow \quad \lambda = \pm i \; ; \text{ imaginary solu.}$$

To deal with this case, we need to explore the notion of complex vector space & related ideas.

VECTORS IN C^n : A vector space in which scalars are allowed to be complex numbers is called a Complex Vector Space. In this Section, we will be concerned only with the following complex generalization of the real vector space R^n.

Definition: If n is a positive integer, then a Complex n-tuple is a sequence of n complex numbers $(v_1, v_2, \cdots, v_n)$. The set of all complex n-tuples is called Complex n-Space and is denoted by C^n. Scalars are complex numbers and the operations of Addition, Subtraction and Scalar multiplication are performed componentwise.

The terminology used for n-tuples of real numbers applies to complex n-tuples without change. Thus, if $v_1, v_2, \cdots, v_n$ are complex numbers, then we call $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ a vector in C^n and $v_1, v_2, \cdots, v_n$ its components.

Some examples of vectors in C^3 are

$$\mathbf{u} = (1+i, -4i, 3+2i), \quad \mathbf{v} = (0, i, 5), \quad \mathbf{w} = \left(6-\sqrt{2}\,i, 9+\tfrac{1}{2}i, \pi i\right).$$

Every vector in C^n can be split into Real and Imaginary part as —

$$\begin{aligned}
\mathbf{v} &= (v_1, v_2, \cdots, v_n) \\
&= (a_1+ib_1, a_2+ib_2, \cdots, a_n+ib_n) \\
&= (a_1, a_2, \cdots, a_n) + i\,(b_1, b_2, \cdots, b_n) \\
&= Re(\mathbf{v}) + i\,Im(\mathbf{v}), \quad \text{where } Re(\mathbf{v}) = (a_1, a_2, \cdots, a_n) \ \& \ Im(\mathbf{v}) = (b_1, b_2, \cdots, b_n)
\end{aligned}$$

The Complex Conjugate of $\mathbf{v}$ can be expressed as —

$$\begin{aligned}
\bar{\mathbf{v}} &= (\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n) \\
&= (a_1-ib_1, a_2-ib_2, \cdots, a_n-ib_n) \\
&= (a_1, a_2, \cdots, a_n) - i\,(b_1, b_2, \cdots, b_n) \\
&= Re(\mathbf{v}) - i\,Im(\mathbf{v})
\end{aligned}$$

It follows that vectors in R^n can be viewed as those vectors in C^n whose imaginary part is zero; or stated another way, a vector $\mathbf{v}$ in C^n is in R^n iff $\bar{\mathbf{v}} = \mathbf{v}$.

In this Section, we will also need to consider matrices with complex entries, so henceforth we will call a matrix A, a Real Matrix if its entries are required to be real numbers and a Complex Matrix if its entries are allowed to be complex numbers.
The standard operations on real matrices carry over to complex matrices without change, and all of the familiar properties of matrices continue to hold.

If A is a Complex Matrix, then $Re(A)$ and $Im(A)$ are the matrices formed from the real and imaginary parts of entries of A, and
$\bar{A}$ is the matrix formed by taking the complex conjugate of each entry in A.

Example ① Real and Imaginary parts of Vectors and Matrices

Let $V = (3+i, -2i, 5)$ in a vector in C^3

then $Re(V) = (3, 0, 5)$

$Im(V) = (1, -2, 0)$

& $\bar{V} = (3-i, 2i, 5)$

Let $A = \begin{bmatrix} 1+i & -i \\ 4 & 6-2i \end{bmatrix}$ is a matrix

then $Re(A) = \begin{bmatrix} 1 & 0 \\ 4 & 6 \end{bmatrix}$

$Im(A) = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$

& $\bar{A} = \begin{bmatrix} 1-i & i \\ 4 & 6+2i \end{bmatrix}$

ALGEBRAIC PROPERTIES OF COMPLEX CONJUGATE

The next two theorems list some properties of complex vectors and matrices that we will need in this Section.

THEOREM ① If u and v are vectors in C^n and if k is a scalar, then

(i) $\bar{\bar{u}} = u$

(ii) $\overline{ku} = \bar{k}\,\bar{u}$

(iii) $\overline{u+v} = \bar{u} + \bar{v}$

(iv) $\overline{u-v} = \bar{u} - \bar{v}$

THEOREM ② If A is an $m \times k$ complex matrix and B is a $k \times n$ complex matrix, then

(i) $\bar{\bar{A}} = A$

(ii) $\overline{(A^T)} = (\bar{A})^T$

(iii) $\overline{AB} = \bar{A}\,\bar{B}$

THE COMPLEX EUCLIDEAN INNER PRODUCT

The following definition extends the notions of dot product and norm to C^n.

Definition: If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ are vectors in C^n, then the Complex Euclidean Inner Product of u and v (also called complex dot product) is denoted by $u.v$ and is defined as

$$u.v = u_1\bar{v}_1 + u_2\bar{v}_2 + \dots + u_n\bar{v}_n \qquad \text{①}$$

We also define the Euclidean Norm on C^n to be

$$\|v\| = \sqrt{v.v}$$
$$= \sqrt{v_1\bar{v}_1 + v_2\bar{v}_2 + \dots + v_n\bar{v}_n}$$
$$\Rightarrow \quad \|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_n|^2} \qquad \text{②}$$

As in the real case, a vector v is called a Unit Vector in C^n if $\|v\| = 1$ and two vectors u and v are said to be Orthogonal if $u.v = 0$.

Example ② Complex Euclidean Inner Product and Norm

Find $u.v$, $v.u$, $\|u\|$ and $\|v\|$ for the vectors $u = (1+i, i, 3-i)$ & $v = (1+i, 2, 4i)$.

Solution:

$$u.v = (1+i)\overline{(1+i)} + i(\bar{2}) + (3-i)\overline{(4i)}$$
$$= (1+i)(1-i) + 2i + (3-i)(-4i) \qquad \left(\because \bar{i} = -i, \ \bar{2} = 2\right.$$
$$= (1)^2 - (i)^2 + 2i - 12i + 4i^2$$
$$= 1 - (-1) - 10i - 4 = -2 - 10i \qquad (\because i^2 = -1)$$

and

$$v.u = (1+i)\overline{(1+i)} + 2\overline{(i)} + 4i\overline{(3-i)}$$
$$= (1+i)(1-i) + 2(-i) + 4i(3+i)$$
$$= (1)^2 - (i)^2 - 2i + 12i + 4i^2 = -2 + 10i$$

$$\|u\| = \sqrt{|1+i|^2 + |i|^2 + |3-i|^2} \qquad \left(\because |a+ib|^2 = a^2 + b^2\right.$$
$$= \sqrt{(1^2+1^2) + (0^2+1^2) + \{(3)^2 + (-1)^2\}}$$
$$\|u\| = \sqrt{2+1+10} = \sqrt{13}$$

and

$$\|v\| = \sqrt{|1+i|^2 + |2|^2 + |4i|^2}$$
$$= \sqrt{\{(1)^2 + (1)^2\} + (2)^2 + \{(0)^2 + (4)^2\}}$$
$$= \sqrt{2+4+16}$$
$$= \sqrt{22}$$

NOTE ① Recall from Sec 3.2 that if u and v are column vectors in R^n, then their dot product can be expressed as

$$u.v = u^T v = v^T u$$

The analogous formulas in C^n are

$$u.v = u^T \bar{v} = \bar{v}^T u$$

NOTE ② Example ② reveals a major difference between the dot product on R^n and the complex dot product on C^n.

For the dot product on R^n, we always have $v.u = u.v$ (Symmetry Property) but for the complex dot product, the corresponding relationship is given by

$u.v = \overline{v.u}$, which is called its Antisymmetry Property.

THEOREM ③ If u, v and w are vectors in C^n and if k is a scalar, then the complex euclidean inner product has the following properties —

(i) $u.v = \overline{v.u}$ [Antisymmetry Property]

(ii) $u.(v+w) = u.v + u.w$ [Distributive Property]

(iii) $k(u.v) = (ku).v$ [Homogeneity Property]

(iv) $u.kv = \bar{k}(u.v)$ [Antihomogeneity Property]

(v) $v.v \geq 0$ and $v.v = 0$ iff $v = 0$ [Positivity Property]

VECTOR CONCEPTS IN C^n: Except for the use of complex scalars, the notions of linear combination, linear independence, subspace, spanning, basis and dimension carry over without change to C^n.

Eigenvalues and Eigenvectors are defined for complex matrices exactly as for real matrices. If A is an $n \times n$ matrix with complex entries, then the complex roots of the characteristic eqn. $\det(\lambda I - A) = 0$ are called Complex eigenvalues of A. As in the real case, λ is a complex eigenvalue of A iff there exists a non-zero vector x in C^n such that $Ax = \lambda x$. Each such x is called a Complex Eigenvector of A corresponding to λ. The complex eigenvectors of A corresponding to λ are the non-zero solutions of the linear system $(\lambda I - A)x = 0$, and the set of all such solutions is a subspace of C^n, called the Eigenspace of A corresponding to λ.

The following theorem states that if a real matrix has complex eigenvalues, then those eigenvalues and their corresponding eigenvectors occur in conjugate pairs.

THEOREM ④ If λ is an eigenvalue of a real $n \times n$ matrix A, and if x is a corresponding eigenvector, then $\bar{\lambda}$ is also an eigenvalue of A and $\bar{x}$ is a corresponding eigenvector.

Example ③ Complex Eigenvalues and Eigenvectors

Find the eigenvalues and bases for the eigenspaces of the matrix $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$.

Soln. The Characteristic Equ. of A is

$$\det(\lambda I - A) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} \lambda+2 & 1 \\ -5 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda-2) - (-5)\times 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 4 + 5 = 0$$

$$\Rightarrow \quad \lambda^2 + 1 = 0 \quad \Rightarrow \lambda = \pm i.$$

So the eigenvalues of A are $\lambda = i$ & $\lambda = -i$. Note that these eigenvalues are complex conjugates, as guaranteed by Theorem ④.

To find the eigenvectors, we must solve the system

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} \lambda+2 & 1 \\ -5 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{——— ①}$$

With $\lambda = i$, the system ① becomes

$$\begin{bmatrix} i+2 & 1 \\ -5 & i-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & i-2 \\ i+2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & \frac{2}{5} - \frac{1}{5}i \\ i+2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_1 \to \left(-\tfrac{1}{5}\right) R_1$$

$$\begin{bmatrix} 1 & \frac{2}{5} - \frac{1}{5}i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \to R_2 - (i+2) R_1$$

$$\Rightarrow \quad x_1 + \left(\tfrac{2}{5} - \tfrac{1}{5}i\right) x_2 = 0 \quad \text{——— (i)}$$

Let $x_2 = t$, then from (i) $\quad x_1 = \left(-\frac{2}{5} + \frac{1}{5}i\right) t$

This tells us that the eigenspace corresponding to $\lambda = i$ is one dimensional and consists of all complex scalar multiples of the basis vector

$$\mathbf{x} = \begin{bmatrix} -\frac{2}{5} + \frac{1}{5}i \\ 1 \end{bmatrix} \quad \text{——— ②}$$

We could find a basis for the eigenspace corresponding to $\lambda = -i$ in a similar way but the work is unnecessary, since Theorem ④ implies that $\bar{\mathbf{x}} = \begin{bmatrix} -\frac{2}{5} - \frac{1}{5}i \\ 1 \end{bmatrix}$ must be a basis for this eigenspace.

THEOREM (5) If A is a 2×2 matrix with real entries, then the characteristic eqn. of A is $\lambda^2 - [\text{trace}(A)]\lambda + \det(A) = 0$ and

(i) A has two distinct real eigenvalues if $[\text{trace}(A)]^2 - 4\det(A) > 0$.

(ii) A has one repeated real eigenvalue if $[\text{trace}(A)]^2 - 4\det(A) = 0$.

(iii) A has two complex conjugate eigenvalues if $[\text{trace}(A)]^2 - 4\det(A) < 0$.

NOTE: From Algebra, if $ax^2+bx+c=0$ is a quadratic eqn. with real coefficients, then the Discriminant b^2-4ac determines the nature of the roots:

$b^2-4ac > 0$ [Two distinct real roots]

$b^2-4ac = 0$ [One repeated real root]

$b^2-4ac < 0$ [Two conjugate imaginary roots]

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

Example (4) Eigenvalues of a 2×2 Matrix

In each part, use the above formula for the characteristic eqn. to find the eigenvalues of

(i) $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

Soln. (i) Here $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$

$$\text{trace}(A) = 2+5 = 7$$

$$\det(A) = (2)(5) - (2)(-1) = 12$$

$\therefore$ The characteristic eqn. of A is $\lambda^2 - 7\lambda + 12 = 0$ $\quad (\because \lambda^2 - [tr(A)]\lambda + \det(A) = 0$

i.e., $(\lambda-4)(\lambda-3) = 0$

$\Rightarrow \lambda = 4$ & $\lambda = 3$.

Soln. (ii) Here $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$$\text{trace}(A) = 0+2 = 2$$

$$\det(A) = (0)(2) - (1)(-1) = 1$$

$\therefore$ The characteristic eqn. of A is $\lambda^2 - [\text{trace}(A)]\lambda + \det(A) = 0$

$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$\Rightarrow (\lambda-1)^2 = 0$

so $\lambda = 1$ is the only eigenvalue of A with algebraic multiplicity 2.

Symmetric Matrices have Real Eigenvalues: Our next result, which is concerned with the eigenvalues of real symmetric matrices, is important in a wide variety of applications.

THEOREM (6) If A is a real symmetric matrix, then A has real eigenvalues.