SEC 2.1 DETERMINANTS BY COFACTOR EXPANSION

In this Section, we will define the notion of a 'Determinant'. This will enable us to give a specific formula for the inverse of an invertible matrix, whereas upto now we have had only a computational procedure for finding it. This, in turn, will eventually provide us with a formula for solutions of certain kinds of linear systems.

Recall that the 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is Invertible iff $ad-bc \neq 0$

and that the expression $ad-bc$ is called the Determinant of the matrix A, denoted by

$$\det(A) = ad-bc$$

or

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \qquad \text{———} \quad (1)$$

and the Inverse of A can be expressed as

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{———} \quad (2)$$

One of our main goals in this Chapter is to obtain an analog of Formula (2) that is applicable to square matrices of all orders.

MINORS AND COFACTORS:

If A is a square matrix, then the Minor of entry a_{ij} is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A, and is denoted by M_{ij}.

The no. $(-1)^{i+j} M_{ij}$ is called the Cofactor of entry a_{ij} and is denoted by C_{ij}.

Example (finding Minors and Cofactors)

Find the Minors and Cofactors of the matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & 6 \\ 3 & 5 & 7 \end{bmatrix}$.

Soln.

The Minor of entry a_{11} (i.e., 1) is $M_{11} = \begin{vmatrix} 0 & 6 \\ 5 & 7 \end{vmatrix} = 0\times7 - 5\times6 = -30$

The Cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = M_{11} = -30$

The Minor of entry a_{12} (i.e., -1) is $M_{12} = \begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix} = 2\times7 - 6\times3 = -4$

The Cofactor of a_{12} is $C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(-4) = 4$.

The Minor of entry a_{13} (i.e., 4) is $M_{13} = \begin{vmatrix} 2 & 0 \\ 3 & 5 \end{vmatrix} = 2\times5 - 3\times0 = 10$

The Cofactor of a_{13} is $C_{13} = (-1)^{1+3} M_{13} = M_{13} = 10$

The Minor of entry a_{21} (i.e., 2) is $M_{21} = \begin{bmatrix} -1 & 4 \\ 5 & 7 \end{bmatrix} = (-1)\times7 - 5\times4 = -27$

The Cofactor of a_{21} is $C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -(-27) = 27$.

The Minor of entry a_{22} (i.e., 0) is $M_{22} = \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = 1\times 7 - 4\times 3 = -5$

The Cofactor of a_{22} is $C_{22} = (-1)^{2+2} M_{22} = M_{22} = -5$.

Similarly, we can find the Minors and Cofactors for other entries.

REMARK: A minor M_{ij} and its corresponding cofactor C_{ij} are either the same or negatives of each other and the relating sign $(-1)^{i+j}$ is either $+1$ or -1 in accordance with the pattern in the 'checkerboard' array

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

for example,

$C_{11} = M_{11}$, $C_{12} = -M_{12}$, $C_{13} = M_{13}, \cdots$

$C_{21} = -M_{21}$, $C_{22} = M_{22}$, $C_{23} = -M_{23}, \cdots$

COFACTOR EXPANSIONS OF a 2X2 MATRIX

Let a 2x2 matrix is $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

The checkerboard pattern for the matrix A is $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$

$\therefore$ $C_{11} = M_{11} = a_{22}$, $C_{12} = -M_{12} = -a_{21}$

$C_{21} = -M_{21} = -a_{12}$, $C_{22} = M_{22} = a_{11}$

Now $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$ ——— ①

$$\left.\begin{aligned} &= a_{11}C_{11} + a_{12}C_{12}, \text{ from ①} \\ &= a_{21}C_{21} + a_{22}C_{22}, \text{ from ①} \\ &= a_{11}C_{11} + a_{21}C_{21}, \text{ from ①} \\ &= a_{12}C_{12} + a_{22}C_{22}, \text{ from ①} \end{aligned}\right\} \text{——— ②}$$

Each equations of ② is called a Cofactor Expansion of $\det(A)$. In each cofactor expansion, the entries and cofactors all come from the same row or same column of A. For example, in the first eqn. of ②, the entries and cofactors all come from first row of A; in the second eqn. of ②, all come from the second row of A; in the third eqn. of ②, all come from the first column of A and in the fourth, all come from second column of A.

DEFINITION OF GENERAL DETERMINANT

THEOREM. If A is an $n \times n$ matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

This result allows us to make the following definition —

Definition: If A is an $n \times n$ matrix, then the no. obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the Determinant of matrix A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{—— (1)}$$

[Cofactor expansion along the j^{th} column]

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{—— (2)}$$

[Cofactor expansion along the i^{th} row]

Example (1) Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ by cofactor expansion along the first row of A.

Soln. $\cancel{\text{The}}$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3[(-4)\times(-2) - 4\times 3] - [(-2)\times(-2) - 5\times 3] + 0$$

$$= 3(-4) - (-11) = -1$$

Example (2) Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$ by cofactor expansion along the first column of A.

Soln.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3[(-4)\times(-2) - 4\times 3] + 2[(-2)\times 1 - 4\times 0] + 5[1\times 3 - 0\times(-4)]$$

$$= 3(-4) + 2(-2) + 5(3) = -1.$$

REMARK: Note that in Example (2), we had to compute three cofactors whereas in Example (1), only two were needed because the third was multiplied by zero.

As a rule, the best strategy for cofactor expansion (to find determinant) is to expand along a row or column with the most zeros.

Example ③ (Smart Choice of Row or Column)

Find the determinant of the matrix, $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$.

Soln. To find det(A), it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

Now for this 3x3 determinant, it will be easiest to use cofactor expansion along its second column, since it has most zeros:

$$\det(A) = 1 \cdot (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2\,[1 \times 1 - 2 \times (-1)]$$

$$= -6.$$

DETERMINANT OF TRIANGULAR MATRIX

If A is an $n \times n$ Triangular matrix (upper triangular, lower triangular or diagonal), then det(A) is the product of entries on the main diagonal of the matrix

i.e., $\det(A) = a_{11}\, a_{22}\, a_{33} \cdots a_{nn}$

Example Find the determinant of the following matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 5 & 1 & 0 & 5 \end{bmatrix}$.

Soln. By above rule, $\det(A) = 2 \times (-3) \times 1 \times 5 = -30$.

SEC-2.2 EVALUATING DETERMINANTS BY ROW REDUCTION

In this Section, we will show how to evaluate a determinant by reducing the associated matrix to row echelon form. In general, this method requires less computation than cofactor expansion and hence is the method of choice for large matrices.

A Basic Theorem. We begin with a fundamental theorem that will lead us to an efficient procedure for evaluating the determinant of a square matrix of any size.

THEOREM: Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Example: The determinants of the following matrices are zero —

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 5 \\ 0 & 1 & 7 \\ 0 & 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 & 0 & 5 \\ 2 & 1 & 0 & 7 \\ 4 & 3 & 0 & 9 \\ 5 & 0 & 0 & 8 \end{bmatrix}$$

THEOREM. Let A be a square matrix. Then $\det(A) = \det(A^T)$.

ELEMENTARY ROW OPERATIONS. — The following theorem shows how an elementary row operation on a square matrix affects the value of its determinant.

THEOREM: Let A is an $n \times n$ matrix

(i) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.

(ii) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.

(iii) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

TABLE 1

RELATIONSHIP	OPERATION
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	The first row of matrix A is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	The first and second rows of 'A' are interchanged
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$	A multiple of second row of 'A' is added to the first row

<u>ELEMENTARY MATRICES</u>. It will be useful to consider the special case of Previous Theorem in which $A = I_n$ is the $n\times n$ identity matrix and E (rather than B) denotes the elementary matrix that results when the row operation is performed on I_n. In this special case, the previous theorem implies the following result.

<u>THEOREM</u>: Let E be an $n\times n$ elementary matrix.

(i) If E results from multiplying a row of I_n by a non-zero no. k, then $\det(E) = k\det(I_n) = k$.

(ii) If E results from interchanging two rows of I_n, then $\det(E) = -\det(I_n) = -1$.

(iii) If E results from adding a multiple of one row of I_n to another, then

$$\det(E) = \det(I_n) = 1.$$

<u>Example</u> (<u>Determinants of Elementary Matrices</u>) – The following determinants of elementary matrices, which are evaluated by inspection, illustrate this theorem –

i) $$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3\det(I_4) = 3\times 1 = 3$$

The second row of I_4 was multiplied by 3.

ii) $$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -\det(I_4) = -1$$

The first and last rows of I_4 were interchanged.

iii) $$\begin{vmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \det(I_4) = 1$$

5 times the last row of I_4 was added to first row of I_4.

MATRICES WITH PROPORTIONAL ROWS OR COLUMNS

THEOREM: If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Proof. If a square matrix A has two proportional rows, then a row of zeros can be introduced by adding a suitable multiple of one of the rows to the other. Similarly for columns. But adding a multiple of one row or column to another does not change the determinant so from previous theorem, we must have $\det(A) = 0$.

Example (Introducing zero Rows) – The following computation shows how to introduce a row of zeros when there are two proportional rows

$$|A| = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix}, \quad R_2 \to R_2 - 2R_1$$

$$= 0$$

Example Find the determinant of the following matrices –

$$A = \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

Soln.

$$|A| = \begin{vmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 7 \\ -4 & 0 & 5 \\ 2 & 0 & 3 \end{vmatrix}, \quad C_2 \to C_2 + 2C_1$$

$$= 0$$

and

$$|B| = \begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad R_4 = R_4 + 3R_1$$

$$= 0$$

EVALUATING DETERMINANTS BY ROW REDUCTION.

We will now give a method for evaluating determinants that involves substantially less computation than cofactor expansion. The idea of the method is to reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of upper triangular matrix (an easy computation) and then relate the determinant to that of the original matrix.

Example (To Evaluate a Determinant Using Row Reduction)

Evaluate det(A), where $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

Soln.

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}, \quad R_1 \leftrightarrow R_2$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix},$$

A common factor of 3 from R_1 is taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}, \quad R_3 \to R_3 - 2R_1$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}, \quad R_3 \to R_3 - 10R_2$$

$$= (-3) \times 1 \times 1 \times (-55)$$

(the det. of triangular matrix is equal to the product of its diagonal elements)

$$= 165$$

SEC-2.3 PROPERTIES OF DETERMINANTS ; CRAMER'S RULE

In this Section, we will develop some fundamental properties of matrices and we will use these results to derive a formula for the inverse of an invertible matrix and formulas for the solutions of certain kinds of linear systems.

BASIC PROPERTIES OF DETERMINANTS

PROP. 1 Let A is an $n\times n$ matrix and k is any scalar then $\det(kA) = k^n \det(A)$

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$

Now $\det(kA) = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$

$= (k)(k)(k) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, take a common factor k from each row through the det. sign

$= k^3 \det(A)$

PROP. 2 Let A and B are $n\times n$ matrices, then usually $\det(A+B) \neq \det(A) + \det(B)$

Example, consider the two matrices of same size as

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} , \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+3 & 2+1 \\ 2+1 & 5+3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}$

$= 5\times 1 - 2\times 2 = 1$ ——— (i)

$\det(B) = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3\times 3 - 1\times 1 = 8$ ——— (ii)

and $\det(A+B) = \begin{vmatrix} 4 & 3 \\ 3 & 8 \end{vmatrix}$

$= 8\times 4 - 3\times 3 = 23$ ——— (iii)

Thus, from (i), (ii) & (iii)

$$\det(A+B) \neq \det(A) + \det(B)$$

NOTE : There is a useful relationship concerning sums of determinants that is applicable when the matrices involved are the same except for one row (column).

For example, consider following two matrices that differ only in the second row:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then
$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \det\begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

This is the special case of following General Result —

THEOREM : Let A, B & C are $n\times n$ matrices that differ only in a single row, say r^{th} and assume that r^{th} row of C can be obtained by adding corresponding entries in the r^{th} rows of A & B. Then
$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

Example : It is left to confirm the following equality by evaluating the determinants

$$\begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{vmatrix} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{vmatrix} + \begin{vmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{vmatrix}$$

PROP. 3 If A and B are square matrices of same size, then $\det(AB) = \det(A).\det(B)$.

Example Verify that $\det(AB) = \det(A).\det B$ by taking two matrices of same size.

Soln. Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$

then $AB = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

Now $\det(A) = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3\times 1 - 2\times 1 = 1$ ——— (i)

$\det(B) = \begin{vmatrix} -1 & 3 \\ 5 & 8 \end{vmatrix} = (-1)\times 8 - 5\times 3 = -23$ ——— (ii)

and $\det(AB) = \begin{vmatrix} 2 & 17 \\ 3 & 14 \end{vmatrix} = 14\times 2 - 17\times 3 = -23$ ——— (iii)

Thus from (i), (ii) & (iii)

$$\det(AB) = \det(A).\det(B)$$

DETERMINANT TEST FOR INVERTIBILITY

THEOREM: A square matrix 'A' is Invertible if and only if $\det(A) \neq 0$.

Example: Show that the following matrix is not Invertible :- $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$.

Soln.

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= 0 \quad \text{, since the first and third rows are proportional.}$$

Thus, the matrix A is not Invertible.

THEOREM. If a matrix A is Invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Proof. We know that

$$A^{-1}A = I$$

$$\therefore \quad \det(A^{-1}A) = \det(I)$$

$$\Rightarrow \det(A^{-1}).\det(A) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \quad \text{, since } \det(A) \neq 0 \text{ if } A \text{ is invertible}$$

ADJOINT OF A MATRIX

The Transpose of the matrix formed by the cofactors of matrix, is called Adjoint of Matrix

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij} then

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$$

$$= \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

__Example__. Find the Adjoint of a matrix A, where

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

__Soln.__

$$C_{11} = \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 6\times 0 - 3\times(-4) = 12$$

$$C_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -\left[1\times 0 - 3\times 2\right] = 6$$

$$C_{13} = \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = (-4)\times 1 - 6\times 2 = -16$$

$$C_{21} = -\begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = -\left[2\times 0 - (-1)\times(-4)\right] = -[0-4] = 4$$

$$C_{22} = \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 3\times 0 - 2\times(-1) = 2$$

$$C_{23} = -\begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = -\left[3\times(-4) - 2\times 2\right] = 16$$

$$C_{31} = \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} = 2\times 3 - 6\times(-1) = 12$$

$$C_{32} = -\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -\left[3\times 3 - 1\times(-1)\right] = -[9+1] = -10$$

$$C_{33} = \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 3\times 6 - 2\times 1 = 16$$

Now matrix of cofactors is $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$ i.e., $\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$

$$\therefore \operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

In Chapter 1, we gave a formula for the inverse of a 2x2 matrix. Our next theorem extends that result to nxn invertible matrices.

THEOREM. (INVERSE OF A MATRIX USING ITS ADJOINT)

If A is an Invertible matrix, then $$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj} A$$

Example : Find the Inverse of the matrix A, where $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$.

Soln.

$$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{vmatrix} = 3\begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix}$$

$$= 3[6\times0 - 3\times(-4)] - 2[1\times0 - 3\times2] - 1[1\times(-4) - 6\times2]$$

$$= 3(12) - 2(-6) - (-16) = 64 \neq 0$$

Hence, the matrix A is Invertible.

Now $\operatorname{adj} A = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$, Do as in previous example again.

$$\therefore A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

$$= \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

CRAMER'S RULE: If $Ax = b$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution.

This solution is $x_1 = \dfrac{\det(A_1)}{\det(A)}$, $x_2 = \dfrac{\det(A_2)}{\det(A)}$, $\ldots$, $A_n = \dfrac{\det(A_n)}{\det(A)}$

Where A_j is matrix obtained by replacing entries in jth column of A by the entries in the matrix b, where $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Example Use Cramer's Rule to solve the following:–

$$\begin{aligned} x_1 \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Soln. Here $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$

$\det(A) = 44$ (calculate)

$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}$, replace first column of matrix A by $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$

$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}$, replace second column of A by b

$A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$, replace third column of A by b

Now $\det(A_1) = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} = 6\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 30 & 6 \\ 8 & 3 \end{vmatrix} + 2\begin{vmatrix} 30 & 4 \\ 8 & -2 \end{vmatrix}$

$= 6(12+12) + 0 + 2(-60-32) = -40$

$\det(A_2) = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix} = 1\begin{vmatrix} 30 & 6 \\ 8 & 3 \end{vmatrix} - 6\begin{vmatrix} -3 & 6 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} -3 & 30 \\ -1 & 8 \end{vmatrix}$

$= 1(90-48) - 6(-9+6) + 2(-24+30) = 72$

$\det(A_3) = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix} = 1\begin{vmatrix} 4 & 30 \\ -2 & 8 \end{vmatrix} - 0\begin{vmatrix} -3 & 30 \\ -1 & 8 \end{vmatrix} + 6\begin{vmatrix} -3 & 4 \\ -1 & -2 \end{vmatrix}$

$= 1(32+60) + 0 + 6(6+4) = 152$

By Cramer's Rule,

$x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{-40}{44} = \dfrac{-10}{11}$; $x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{72}{44} = \dfrac{18}{11}$

and $x_3 = \dfrac{\det(A_3)}{\det(A)} = \dfrac{152}{44} = \dfrac{38}{11}$

$$\boxed{\det A = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = 44}$$