Chapter-2.

Sets, Functions, Sequences, Sums and Matrices

Week-5.

Set:- A set is an unordered collection of objects. These objects are called elements or members of set.

Sets are usually denoted by capital letters.
Elements are denoted by small letters.

Eg $A = \{a, b, c, d\}$ — Roster form.

Here $a \in A$ and $x \notin A$.

$A = \{x / x \in \mathbb{Z}^+ \text{ and } x < 100\}$ — Set builder form.

$= \{1, 2, \ldots, 99\}$.

Intervals

$[a, b] = \{x / a \leq x \leq b\}$ (Closed interval)

$[a, b) = \{x / a \leq x < b\}$ (Semi-closed (or) closed at 'a')

$(a, b] = \{x / a < x \leq b\}$

$(a, b) = \{x / a < x < b\}$ (open interval)

Equality of Sets:- Two sets A and B are said to be equal if and only if they have same elements. we write this as $A = B$

Eg $A = \{1, 3, 5\}$; $B = \{3, 5, 1\}$; $C = \{1, 3, 3, 5, 5, 5\}$

Here $A = B = C$.

Mathematically equality means $\forall x (x \in A \leftrightarrow x \in B)$

Empty Set (or) Null Set (or) Void Set:-

A set having no elements is called a null set or empty set. This is denoted by $\emptyset$ and $\emptyset = \{ \}$

Singleton Set:- A set having only one element

Eg

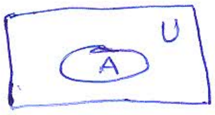
$A = \{a\}$

$B = \{\{ \}\}$ → Here the element is null set.

or

$B = \{\emptyset\}$

Venn Diagram:- The graphical representation of set.



Here U is called universal set which contains all objects under consideration. This is represented by rectangle

Subset:- The set A is a subset of B iff every element of A is also an element of B. We write this as $A \subseteq B$

Mathematically $\forall x\,(x \in A \rightarrow x \in B)$ is true

Eg

A = Set of all positive integers less than $10 = \{1,3,5,7,9\}$

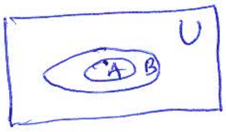
$B = \{1,2,3,4,5,6,7,8,9\}$ Set of all integers less than 10

Here $A \subseteq B$

Proper Subset: If A is a subset of B and $A \neq B$ then A is a proper subset of B. We write this as $A \subset B$. That means there must be atleast one element in B which is not in A. Mathematically $A \subset B$ means

$$\forall x\,(x \in A \rightarrow x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$$

Venn diagrams can be used to show proper subsets.



Here $A \subset B \subset U$.

NOTE:-

For every set S

(i) $\phi \subset S$

(ii) $S \subseteq S$

Example Subsets of $A = \{a, b\}$ are $\{a\}, \{b\}, \{a,b\}, \phi,$

NOTE:- If two sets A and B are equal i.e. $A = B$ then $A \subseteq B$ and $B \subseteq A$

Cardinality of Set (Size of Set):

The number of distinct elements in a set S is called as cardinality. This is denoted by $|S|$

Ex 1) $A = \{1, 3, 5, 7, 9\}$

$|A| = 5$

2) $S = \{$ set of all english alphabets $\}$

$|S| = 26$.

3) $\phi = \{\ \}$

$|\phi| = 0$

Power Set:- The power set of a set S is the set of all subsets of S. This is denoted by P(S)

Eg ① $S = \{a, b, c\}$

$P(S) = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}$
$\{a,b,c\}, \emptyset, \underset{\{a,b,c\}}{S}\}$.

NOTE If a set has n elements, then the power set has 2^n elements (subsets)

Eg ② If $\emptyset = \{\ \}$
then $P(\emptyset) = \{\emptyset = \{\ \}\}$ (The empty set has exactly one subset, namely itself).

Eg ③ If $S = \{\{\emptyset\}\}$
then $P(S) = \{\emptyset, \{\emptyset\}\}$.

Cartesian Product:- The cartesian product of two sets A and B is denoted by $A \times B$ and is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$

Eg $A = \{a, b, c\}$; $B = \{p, q, r\}$
$A \times B = \{(a,p), (a,q), (a,r), (b,p), (b,q), (b,r)$
$(c,p), (c,q), (c,r)\}$

Eg $A = \{1, 2\}$; $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

Note that $A \times B \neq B \times A$

and $A \times B = B \times A$ if $A = \phi$ (or) $B = \phi$
then that $A \times B = \phi$

Eg If $A = \{0, 1\}$, $B = (1, 2)$, $C = \{0, 1, 2\}$

$A \times B = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$

$$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}.$$

Eg If $A = \{1, 2\}$

$A^2 = A \times A = \{1, 2\} \times \{1, 2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

$A^3 = A^2 \times A = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$

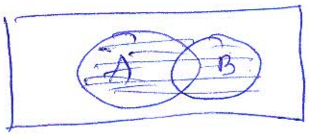
Eg Write the ordered pairs (a, b) if $a < b$ on the set $\{0, 1, 2, 3\}$

Solution $\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$.

Set operations:-

UNION:- $A \cup B = \{x \mid x \in A \vee x \in B\}$.

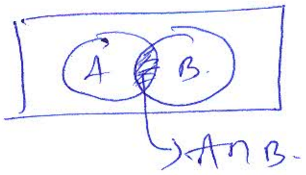
Eg $A = \{1, 3, 5\}$, $B = \{1, 2, 3\}$



$A \cup B = \{1, 2, 3, 5\}$.

Intersection:- $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Eg $A = \{1, 3, 5\}$, $B = \{1, 2, 3\}$



$A \cap B = \{1, 3\}$.

Disjoint Sets:- Two sets are disjoint if their intersection is an empty set

i.e. $A \cap B = \emptyset$

Eg $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7\}$

$A \cap B = \emptyset$

NOTE:- $|A \cup B| = |A| + |B| - |A \cap B|$

(Cardinality of union).

Eg $A = \{1, 2, 5\}$, $B = \{1, 2, 3\}$

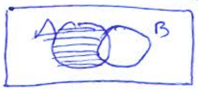
$A \cup B = \{1, 2, 3, 5\}$, $A \cap B = \{1, 2\}$

$|A| = 3$, $|B| = 3$, $|A \cup B| = 4$, $|A \cap B| = 2$

$|A \cup B| = |A| + |B| - |A \cap B|$ $(\because 4 = 3 + 3 - 2)$

Difference of Sets

$A - B = \{ x / x \in A \wedge x \notin B \}$.



Eg $A = \{1, 2, 3, 4\}$ $B = \{2, 4\}$

$A - B = \{1, 3\}$.

Compliment of a Set.

$\bar{A} = \{ x \in U / x \notin A \}$ or $U - A$

Eg If $U = \{1, 2, 3, 4, 5, 6\}$.
and $A = \{1, 2, 5\}$
then $\bar{A} = \{3, 4, 6\}$

Identities

1) $A \cap U = A$
$A \cup \emptyset = A$

2) $A \cup U = U$
$A \cap \emptyset = \emptyset$

3) $A \cup A = A$
$A \cap A = A$

4) $\overline{(\bar{A})} = A$

5) $A \cup B = B \cup A$
$A \cap B = B \cap A$

6) $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

7) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8) DEMORGAN'S LAW
$\overline{A \cap B} = \bar{A} \cup \bar{B}$
$\overline{A \cup B} = \bar{A} \cap \bar{B}$

9) $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$

10) $A \cup \bar{A} = U$
$A \cap \bar{A} = \emptyset$

Proving Set Identities using membership tables

If an element is in set, then 1 is used to indicate it.
If an element is not in set, then 0 is used to indicate it.

Eg:- Use a membership table and show that

(i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution

A	B	C	$(A \cup B)$	$(B \cup C)$	$A \cap B$	$A \cap C$	$A \cup (B \cup C)$	$(A \cup B) \cup C$	$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1	1	1	1
1	0	1	1	1	0	1	1	1	1	1
1	0	0	1	0	0	0	1	1	0	0
0	1	1	1	1	0	0	1	1	0	0
0	1	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	0	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0

From membership table it can be observed that

$A \cup (B \cup C) = (A \cup B) \cup C$.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

NOTE $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$

Computer representation of sets.

Let $U = \{1,2,3,4,5,6,7,8,9,10\}$.

Bit string representing all odd integers in U.
i.e, $\{1,3,5,7,9\}$ is 1010101010

Even numbers in U $\{2,4,6,8,10\}$ is 0101010101

Integers that do not exceed 5 i.e, $\{1,2,3,4,5\}$ is 1111100000

NOTE

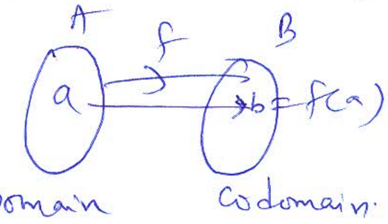
Compliment of 1101010011 is 0010101100

1111100000 $\vee$ 1010101010 = 1111101010
$\{1,2,3,4,5\} \cup \{1,3,5,7,9\} = \{1,2,3,4,5,7,9\}$

1111100000 $\wedge$ 1010101010 = 1010100000
$\{1,2,3,4,5\} \cap \{1,3,5,7,9\} = \{1,3,5\}$

FUNCTIONS:

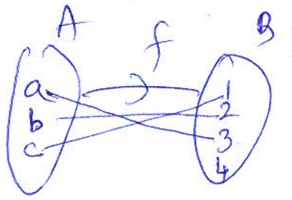
If A and B are non-empty sets, then a function from A to B is denoted by $f: A \to B$ and is defined as assigning <u>exactly one</u> element of B to every element of A i.e., $f(a) = b$



NOTE

If $f(a) = b$, then b is called as the image of a and a is preimage of b.

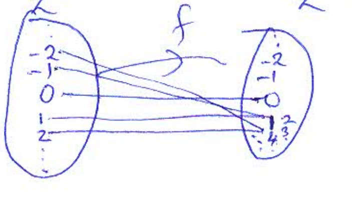
Range of function:- Set of all images of elements of domain



Domain $= \{a, b, c\}$
Codomain $= \{1, 2, 3, 4\}$
Range $= \{1, 2, 3\}$.

Example $f: \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x^2$

Domain $= \mathbb{Z}$
Codomain $= \mathbb{Z}$
Range $= 0, 1, 4, 9, 16, 25, \ldots$



Equality of functions

Two functions f and g are said to be equal if they have same domain, same Codomain and $f(x) = g(x)$ $\forall x$ in domain (common)

NOTE

1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
2) $(f_1 f_2)(x) = f_1(x) f_2(x)$

Eg $f_1(x) = x^3$ and $f_2(x) = x - x^3$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$= x^3 + x - x^3$$
$$= x$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$
$$= x^3 (x - x^3)$$
$$= x^4 - x^6$$

One-one function (or) Injection:-

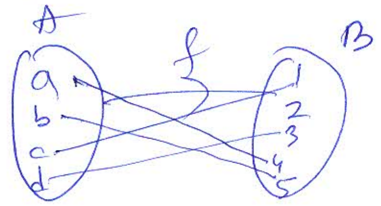
A function $f: A \to B$ is said to be one-one or injection if and only if $f(a) = f(b)$ $\Rightarrow a = b$, $\forall a, b \in A$

(or) iff $f(a) \neq f(b)$ whenever $a \neq b$.

(or) iff $a \neq b \Rightarrow f(a) \neq f(b)$

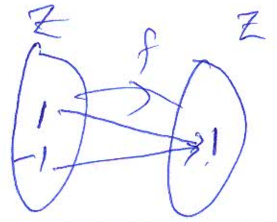
Examples

1). $f: A \to B$ where $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4, 5\}$
with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, $f(d) = 3$.



f is one - one.

2) Show that $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$
is not one - one.



Solution Let $a, b \in \mathbb{Z}$.
and let $f(a) = f(b)$

$\Rightarrow a^2 = b^2$ ($\because f(x) = x^2$)

$\Rightarrow a = \pm b$

$\Rightarrow f$ is not one - one.

For example
Let $1, -1 \in \mathbb{Z}$
$f(1) = (1)^2 = 1$
$f(-1) = (-1)^2 = 1$
$\Rightarrow f(1) = f(-1)$.
but $1 \neq -1$

3) Show that $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^3$ is one-one

Solution Let $a, b \in \mathbb{Z}$ and $a \neq b$.

$\Rightarrow a^3 \neq b^3$

$\Rightarrow f(a) \neq f(b)$

$\Rightarrow f$ is one - one

4) Show that $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x+1$ is one - one

Sol let $a, b \in \mathbb{R}$ and let $a \neq b$

$\Rightarrow a+1 \neq b+1$

$\Rightarrow f(a) \neq f(b)$

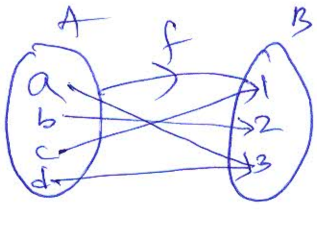
$\Rightarrow f$ is one - one.

Onto function (or) Surjection :-

A function $f: A \to B$ is said to be onto or surjection iff $f(a) = b$, $\forall\, b \in B$

means Codomain = Range.

Example 1) $f: A \to B$ where $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is f is onto ?



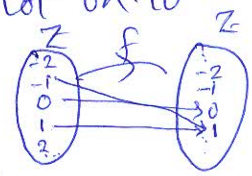
Solution Codomain = Range.

$\Rightarrow f$ is onto.

2) $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$ is not onto.

Reason: Codomain $\neq$ Range

as $f(x) = x^2$ is Positive $\forall x \in \mathbb{Z}$.



3) $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x+1$ is onto.

Sol For every $b \in \mathbb{Z}$ (Codomain) there is atleast one $a = b-1$ (domain) such that $f(a) = b$.

$\Rightarrow f$ is onto.

One-one onto function (or) Bijection:-

A function $f: A \to B$ which is both one-one and onto is called a bijection.

Identity function:

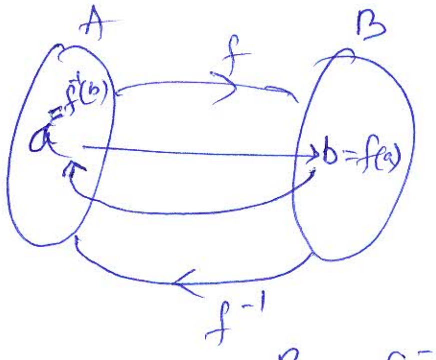
$$I_A : A \to A \quad \text{where } I_A(x) = x$$

Means every element in domain is related to its own element in codomain

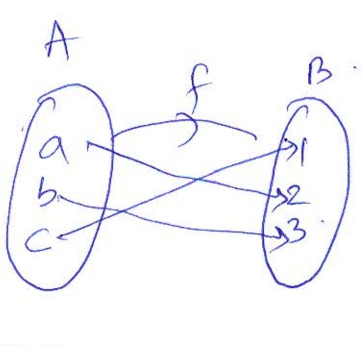
NOTE:- Identity function is bijection.

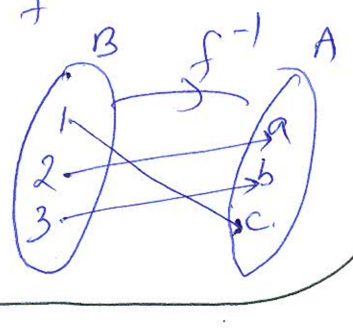
Inverse function:-

If $f: A \to B$ is a bijective function, then its inverse is a function from $B \to A$ and is denoted by $f^{-1}: B \to A$



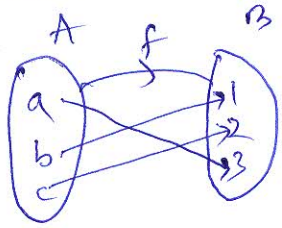
Eg





<u>Examples</u>:- Let $f : A \to B$ where $A = \{a, b, c\}$
$B = \{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 1$
and $f(c) = 2$. Is f invertible? If so, what
is its inverse.

<u>Sol</u>



clearly f is one-one and f is onto.
$\Rightarrow f$ is bijection
$\Rightarrow f^{-1}$ exists. (means f is invertible)
So $f^{-1}(1) = b$, $f^{-1}(2) = c$, $f^{-1}(3) = a$.

(2). $f : Z \to Z$ defined by $f(x) = x + 1$. If f invertible?
If so, what is its inverse.

<u>Sol</u> Let $a, b \in Z$ (domain)
and let $a = b$
$\Rightarrow a + 1 = b + 1$
$\Rightarrow f(a) = f(b)$
$\Rightarrow f$ is one-one.

<u>Now</u>
$f(a) = b$
$\Rightarrow a + 1 = b$
$\Rightarrow a = b - 1$
$\Rightarrow f^{-1}(b) = b - 1$

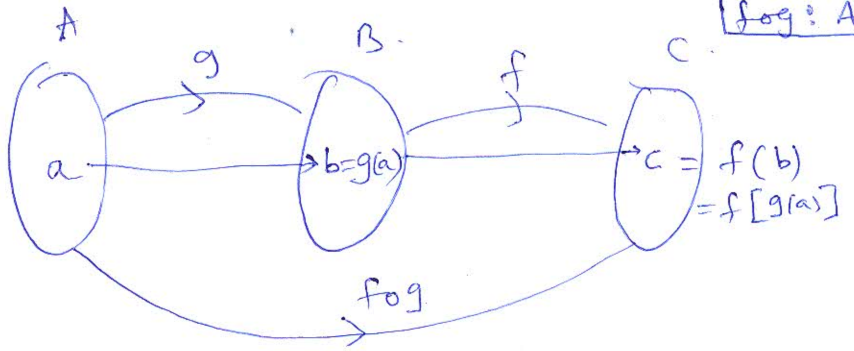
For every $b \in Z$ (codomain) there is atleast
one $a = b - 1 \in Z$ (domain) such that $f(a) = b$
$\Rightarrow f$ is on-to.
As f is both one-one and onto, therefore
$f : Z \to Z$ is bijection and f is invertible.
$\Rightarrow f^{-1} : Z \to Z$ and $f^{-1}(x) = x - 1$

Composition of functions :-

If $g : A \to B$ and $f : B \to C$ are two functions, then the composition of f and g is denoted by fog and $fog : A \to C$ defined as.

$$(fog)(a) = f[g(a)] \;, \forall\, a \in A$$

$g : A \to B$
$f : B \to C$
$fog : A \to C$



NOTE:- In order to define the composition of f and g, the codomain of g must be equal to domain of f.

Examples Let $g : A \to A$ and $f : A \to C$, where $A = \{a, b, c\}$ and $C = \{1, 2, 3\}$ such that $g(a) = b$, $g(b) = c$, $g(c) = a$, $f(a) = 3$, $f(b) = 2$, $f(c) = 1$

What is the composition of f and g?
What is the composition of g and f?

Sol $g : A \to A$
$f : A \to C$

$\Rightarrow fog : A \to C$.

Now $(fog)(a) = f(g(a)) = f(b) = 2$
$(fog)(b) = f(g(b)) = f(c) = 1$
$(fog)(c) = f(g(c)) = f(a) = 3$

$f : A \to C$.
$g : A \to A$.

$\Rightarrow gof$ is not defined as the codomain of f is not equal to domain of g.

② $f: Z \to Z$ and $g: Z \to Z$ are two functions defined by $f(x) = 2x+3$ and $g(x) = 3x+2$.
What is the composition $g \cdot f$ and g?
What is the composition of g and f!

Sol. $g: Z \to Z$
$f: Z \to Z$

$\Rightarrow f \circ g: Z \to Z$.

Now $(f \circ g)(x) = f(g(x))$
$= f(3x+2)$
$= 2(3x+2)+3$
$= 6x+7$

$f: Z \to Z$
$g: Z \to Z$

$\Rightarrow g \circ f: Z \to Z$

Now $(g \circ f)(x) = g(f(x))$
$= g(2x+3)$
$= 3(2x+3)+2$
$= 6x+11$

③ If $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2+1$ and $g(x) = x+2$. Find $f \circ g$ & $g \circ f$

Sol $f: \mathbb{R} \to \mathbb{R}$
$g: \mathbb{R} \to \mathbb{R}$

$\Rightarrow g \circ f: \mathbb{R} \to \mathbb{R}$

$(g \circ f)(x) = g(f(x))$
$= g(x^2+1)$
$= (x^2+1)+2$
$= x^2+3$.

$g: \mathbb{R} \to \mathbb{R}$
$f: \mathbb{R} \to \mathbb{R}$
$f \circ g: \mathbb{R} \to \mathbb{R}$

$(f \circ g)(x) = f(g(x))$
$= f(x+2)$
$= (x+2)^2+1$
$= x^2+4x+5$

Some Important functions:-

Floor function

$\lfloor x \rfloor$ = Integer to left of x.

Eg $\lfloor \frac{1}{2} \rfloor = 0$.

$\lfloor -\frac{1}{2} \rfloor = -1$

$\lfloor 3.1 \rfloor = 3$.

$\lfloor 5 \rfloor = 5$

Ceiling function.

$\lceil x \rceil$ = Integer to right of x.

Eg. $\lceil \frac{1}{2} \rceil = 1$

$\lceil -\frac{1}{2} \rceil = 0$

$\lceil 3.1 \rceil = 4$

$\lceil 5 \rceil = 5$

SEQUENCE:- A sequence is a function from set of integers. (Preferably $\{0, 1, 2, \ldots\}$ (or) $\{1, 2, \ldots\}$) to a set S.

NOTE: a_n is used to denote the image of n. Where a_n is a term of sequence.

Examples 1) The sequence $\{a_n\}$ where $a_n = \frac{1}{n}$. is $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots$

2) The sequence $\{a_n\}$ where $a_n = (-1)^n$. is $1, -1, 1, -1, \ldots$

(3) The geometric progression is a sequence of the form $a, ar, ar^2, \ldots ar^{n-1}, \ldots$ where a is initial term and r is common ratio.

For eg: $1, 3, 3^2, 3^3, \ldots 3^{n-1}, \ldots$

Here $a=1$, $r=3$.

$6, 6(\frac{1}{3}), 6(\frac{1}{3})^2, \ldots 6(\frac{1}{3})^{n-1}$

Here $a=6$, $r=1/3$.

(4) The arithmetic progression is a sequence of the form $a, a+d, a+2d, \ldots, a+(n-1)d, \ldots$ where a is initial term and d is common difference.

For eg. $1, 2, 3, 4, \ldots$

Here $a=1$, $d=1$

$3, 5, 7, 9, \ldots$

Here $a=3$, $d=2$

RECURRENCE RELATIONS :- A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

Example 1) $a_n = a_{n-1} + 3$. for $n = 1, 2, \ldots$

2) $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, \ldots$

Problems (1). The Fibonacci sequence is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0, f_1 = 1$. Find f_2, f_3, f_4, f_5 and f_6.

Sol . Given $f_n = f_{n-1} + f_{n-2}$ — (1) ; $f_0 = 0, f_1 = 1$

Put $n = 2$ in (1) $\Rightarrow$ $f_2 = f_1 + f_0$
$= 1 + 0$
$= 1$

Put $n = 3$ in (1) $\Rightarrow$ $f_3 = f_2 + f_1$
$= 1 + 1$
$= 2$

Put $n = 4$ in (1) $\Rightarrow$ $f_4 = f_3 + f_2$
$= 2 + 1$
$= 3$

Put $n = 5$ in (1) $\Rightarrow$ $f_5 = f_4 + f_3$
$= 3 + 2$
$= 5$

Put $n = 6$ in (1) $\Rightarrow$ $f_6 = f_5 + f_4$
$= 5 + 3$
$= 8$.

NOTE: A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

(2) If $a_n = 6a_{n-1}$, $a_0 = 2$ find a_1, a_2, a_3

Sol Given $a_n = 6a_{n-1}$ — (1) ; $a_0 = 2$

Put $n=1$ in (1) $\Rightarrow a_1 = 6a_0 = 6(2) = 12$

Put $n=2$ in (1) $\Rightarrow a_2 = 6a_1 = 6(12) = 72$

Put $n=3$ in (1) $\Rightarrow a_3 = 6a_2 = 6(72) = 432$

(3) If $a_n = 2^n + 5\cdot 3^n$. Find a_0, a_1, a_2

Sol Given $a_n = 2^n + 5\cdot 3^n$ — (1)

Put $n=0$ in (1) $\Rightarrow a_0 = 2^0 + 5\cdot 3^0 = 1 + 5(1) = 6$

Put $n=1$ in (1) $\Rightarrow a_1 = 2^1 + 5\cdot 3^1 = 2+15 = 17$

Put $n=2$ in (1) $\Rightarrow a_2 = 2^2 + 5\cdot 3^2 = 4+45 = 49$

(4) If $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$ find a_2, a_3 and a_4

Sol Given $a_n = a_{n-1} + 3a_{n-2}$ — (1) ; $a_0 = 1$, $a_1 = 2$

Put $n=2$ in (1) $\Rightarrow a_2 = a_1 + 3a_0 = 2 + 3(1) = 5$

Put $n=3$ in (1) $\Rightarrow a_3 = a_2 + 3a_1 = 5 + 3(2) = 11$

Put $n=4$ in (1) $\Rightarrow a_4 = a_3 + 3a_2 = 11 + 3(5) = 26.$

SUMMATIONS:- Addition of the terms of a Sequence.

$$\sum_{m=1}^{n} a_m = a_1 + a_2 + a_3 + \text{---} + a_n.$$

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

Examples

1) $\displaystyle\sum_{j=1}^{100} \frac{1}{j} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} + \frac{1}{100}.$

2) What is the value of $\displaystyle\sum_{i=1}^{5} i^2$

Sol $\displaystyle\sum_{i=1}^{5} i^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2$
$= 1 + 4 + 9 + 16 + 25$
$= 55.$

3) What is the value of $\displaystyle\sum_{i=4}^{8} (-1)^i$

Sol $\displaystyle\sum_{i=4}^{8} (-1)^i = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$
$= 1 - 1 + 1 - 1 + 1$
$= 1$

4) What is the value of $\displaystyle\sum_{i=1}^{4} \sum_{j=1}^{3} i j$

Sol $\displaystyle\sum_{i=1}^{4} \sum_{j=1}^{3} i j = \sum_{i=1}^{4} (i + 2i + 3i)$

$$= \sum_{i=1}^{4} 6\,i$$

$$= 6(1) + 6(2) + 6(3) + 6(4)$$

$$= 6 + 12 + 18 + 24$$

$$= 60$$

5) $\displaystyle\sum_{K=1}^{n} K = 1 + 2 + 3 + \text{---} + n.$

$$= \frac{n(n+1)}{2}$$

6) $\displaystyle\sum_{K=1}^{n} K^2 = 1^2 + 2^2 + 3^2 + - + n^2$

$$= \frac{n(n+1)(2n+1)}{6}$$

7) $\displaystyle\sum_{K=1}^{n} K^3 = 1^3 + 2^3 + 3^3 + \text{---} + n^3$

$$= \frac{n^2(n+1)^2}{4}$$

8) Find $\displaystyle\sum_{K=50}^{100} K^2$

Sol: We know that $\displaystyle\sum_{K=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$

$$\Rightarrow \sum_{K=1}^{100} K^2 = \frac{100 \times 101 \times 201}{6} = 338350$$

$$\Rightarrow \sum_{K=1}^{49} K^2 + \sum_{K=50}^{100} K^2 = 338350$$

$$\Rightarrow \sum_{K=50}^{100} K^2 = 338350 - \sum_{K=1}^{49} K^2$$

$$= 338350 - \frac{49 \times 50 \times 99}{6}$$

$$= 338350 - 40425$$

$$\sum_{k=50}^{100} k^2 = 297925$$

9) Find $\sum_{j \in S} j^2$ where $S = \{1, 3, 5, 7\}$

Sol $\sum_{j \in S} j^2 = \sum_{j=1,3,5,7} j^2$

$$= (1)^2 + (3)^2 + (5)^2 + (7)^2 = 1 + 9 + 25 + 49$$
$$= 84$$

10) Find $\sum_{j=1}^{8} 2^j$

Sol $\sum_{j=1}^{8} 2^j = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8$

$$= 2(1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7) = 2\frac{(2^8 - 1)}{2-1} = 510$$

NOTE $\sum_{k=0}^{n} ar^k = \begin{cases} \frac{a(r^{n+1} - 1)}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$

11) Find $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j)$

Sol $\sum_{i=0}^{3} \sum_{j=0}^{2} (3i + 2j) = \sum_{i=0}^{3} (3i + 2(0) + 2(1) + 2(2))$

$$= \sum_{i=0}^{3} (3i + 6)$$

$$= (3(0) + 6) + (3(1) + 6) + (3(2) + 6) + (3(3) + 6)$$
$$= 6 + 9 + 12 + 15$$
$$= 42$$

12) Find $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3$

Sol $\sum_{i=0}^{2} \sum_{j=0}^{3} i^2 j^3 = \sum_{i=0}^{2} i^2 (0^3+1^3+2^3+3^3)$

$= \sum_{i=0}^{2} i^2 (36)$

$= 0^2(36) + 1^2(36)$

$= 36.$

MATRICES:- A matrix is an rectangular array of numbers.

Eg $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$

The horizonal lines are called ROWS and vertical lines of elements are called COLUMNS.

A matrix with m rows and n columns is called $m \times n$ matrix.

NOTE:- Matrices are denoted by Capital letters.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

a_{ij} means element in i^{th} row and j^{th} column (or) $A = [a_{ij}]_{m \times n}$

Sum of Matrices:-

If $A = [a_{ij}]_{m \times n}$; $B = [b_{ij}]_{m \times n}$ are two matrices of same size (or) type then their sum is $A + B = [a_{ij} + b_{ij}]_{m \times n}$. is obtained by adding corresponding elements of A and B

Ex:
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3} ; \quad B = \begin{bmatrix} -2 & -3 & 4 \\ 7 & -8 & 9 \\ 4 & 2 & 6 \end{bmatrix}_{3 \times 3}$$

$$A + B = \begin{bmatrix} 1-2 & 2-3 & 3+4 \\ 4+7 & 5-8 & 6+9 \\ 7+4 & 8+2 & 9+6 \end{bmatrix}_{3 \times 3}.$$

$$= \begin{bmatrix} -1 & -1 & 7 \\ 11 & -3 & 15 \\ 11 & 10 & 15 \end{bmatrix}_{3 \times 3}$$

Product of Matrices:-

The necessary condition for multiplying A and B is that number of columns in A must be equal to numbers of rows in B.

If A is $m \times k$ type matrix, B is $k \times n$ type matrix then AB is defined and AB is $m \times n$ type matrix.

$$[A]_{m \times k} [B]_{k \times n} = [AB]_{m \times n}.$$

If $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$ and $B = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$ then $AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_n \\ R_2C_1 & R_2C_2 & \cdots & R_2C_n \\ R_mC_1 & R_mC_2 & \cdots & R_mC_n \end{bmatrix}_{m \times n}$

Example $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}_{4\times 3}$ $\quad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2}$.

$$AB = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}_{4\times 3} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} (1\ 0\ 4)(2\ 1\ 3) & (1\ 0\ 4)(4\ 1\ 0) \\ (2\ 1\ 1)(2\ 1\ 3) & (2\ 1\ 1)(4\ 1\ 0) \\ (3\ 1\ 0)(2\ 1\ 3) & (3\ 1\ 0)(4\ 1\ 0) \\ (0\ 2\ 2)(2\ 1\ 3) & (0\ 2\ 2)(4\ 1\ 0) \end{bmatrix}_{4\times 2}$$

$$= \begin{bmatrix} 2+0+12 & 4+0+0 \\ 4+1+3 & 8+1+0 \\ 6+1+0 & 12+1+0 \\ 0+2+6 & 0+2+0 \end{bmatrix}_{4\times 2}$$

$$= \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}_{4\times 2}$$

Transpose of a Matrix :- The matrix obtained by interchanging rows and colums.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Symmetric Matrix :- A square matrix A is said to be symmetric if $A^T = A$

Eg If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$\Rightarrow A^T = A$.

Eg $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$\Rightarrow A = A^T \Rightarrow A$ is Symmetric.

Zero-one Matrices:- Matrix with all entries either 0 or 1

Eg $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Note 1) $1 \wedge 1 = 1$; $1 \wedge 0 = 0$; $0 \wedge 1 = 0$; $0 \wedge 0 = 0$

$1 \vee 1 = 1$; $1 \vee 0 = 1$; $0 \vee 1 = 1$; $0 \vee 0 = 0$

2) Join of two Zero-one matrices $A \vee B$.

Meet of two Zero-one Matrices $A \wedge B$.

Eg Find the Meet and join of Matrices $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Sol Meet of matrices $A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Join of matrices $A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Boolean Product of Zero-one Matrices $A \odot B$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$A \odot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3\times 2} \odot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} (1\ 0)(1\ 0) & (1\ 0)(1\cdot 1) & (1\ 0)(0\ 1) \\ (0\ 1)(1\ 0) & (0\ 1)(1\ 1) & (0\ 1)(0\ 1) \\ (1\ 0)(0\ 1) & (1\ 0)(1\ 1) & (1\ 0)(0\ 1) \end{bmatrix}$$

$$= \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 0)\vee(0\wedge 1) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1\vee 0 & 1\vee 0 & 0\vee 0 \\ 0\vee 0 & 0\vee 1 & 0\vee 1 \\ 0\vee 0 & 1\vee 0 & 0\vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$