Chapter-1 (Summary)

Logic and Proofs - II

Week - 4

PREDICATE:- Consider the statement "$x > 3$"
when the value of x is not specified, then the statement is neither true nor false.

Here x is called as variable and "is greater than 3" is called as Predicate (subject of statement can have). This is denoted by $P(x)$ where x is variable and P stand for the predicate "is greater than 3".

NOTE:- $P(x)$ is a statement and once a value is assigned to variable x, ~~it~~ becomes proposition and has a truth value.

Eg 1 Let $P(x)$ be the statement "$x > 3$"
then $P(4)$ is True. ($\because 4 > 3$)
$P(2)$ is False ($\because 2 \not> 3$).

② Let $P(x,y)$ be the statement "$x = y+3$"
then $P(1,2)$ is False ($\because 1 \neq 2+3$)
$P(3,0)$ is True ($\because 3 = 0+3$).

③ Let $P(x,y,z)$ be the statement "$x+y=z$"
then $P(1,2,3)$ is True ($\because 1+2=3$)
$P(0,0,1)$ is False ($\because 0+0 \neq 1$)

QUANTIFIERS:- The statement $P(x)$ becomes a proposition when x is assigned a value and has certain truth value.
If $P(x)$ is true/false for all values of x?
If $P(x)$ is true/false for atleast one value of x?

QUANTIFICATION expresses the extent to which a predicate is true over a range of elements

UNIVERSAL Quantifier:-

The universal quantification of $P(x)$ is the statement " $P(x)$ for all values of x in the domain"
This is denoted by $\forall x\, P(x)$ read as "for all x $P(x)$" (or) "for every x $P(x)$".
Here $\forall$ is called the universal quantifier.

Eg ① Let $P(x)$ be the statement " $x+1 > x$ " what is the truth value of the quantification $\forall x P(x)$, when the domain is set of all real numbers.

Ans $\forall x P(x)$ is true (∵ For any real number x, $x+1 > x$.)

② Let $P(x)$ be the statement " $x < 2$ " What is the truth value of the quantification $\forall x P(x)$ where the domain is set of all real numbers.

Ans $\forall x P(x)$ is False (∵ For any real number x, $x \not< 2$ (Eg $x=4 \Rightarrow 4 \not< 2$)

③ Let $P(x)$ be " $x^2 > 0$ " and the domain is set of all integers.

Ans $\forall x P(x)$ is False (∵ $x=-1$ is an integer $x^2=(-1)^2=1>0$ True
$x=0$ is an integer $x^2=0^2=0 \not> 0$ False

④ $P(x)$ is " $x^2 < 10$ " domain is positive integers not exceeding 4.
Ans $\forall x P(x)$ is False (Since domain $x=1,2,3,4$ $1^2<10$, $2^2<10$, $3^2<10$, $4^2 \not< 10$)

⑤ $P(x)$ is $x^2 \geq x$ domain is all real numbers.
Ans $\forall x P(x)$ is False (Since domain x is all real nos Let $x=0.5$; $x^2=(0.5)^2=0.25 \not\geq x=0.5$
When domain is integers than $\forall x P(x)$ is true.

Existential Quantifier

The existential quantification of the statement $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$".
This is denoted by $\exists x\, P(x)$. Here $\exists$ is called the existential quantifier.

NOTE: $\exists x\, P(x)$ means

There is an x such that $P(x)$

(or)

There is atleast one x such that $P(x)$

(or)

For some x $P(x)$

NOTE:- $\forall x\, P(x)$ $\xrightarrow{\text{True}}$ $P(x)$ is true for every x.

$\xrightarrow{\text{False}}$ There is an x for which $P(x)$ is false

$\exists x\, P(x)$ $\xrightarrow{\text{True}}$ There is an x for which $P(x)$ is true

$\xrightarrow{\text{False}}$ There is no x for which $P(x)$ is true i.e., $P(x)$ is false for every x.

Examples (1) Let $P(x)$ be the statement "$x \geq 2$" where $x \in \mathbb{R}$, then $\exists x\, P(x)$ is true

Reason:- There is an x say $3 \in \mathbb{R}$ such that $3>2$

(2) Let $P(x)$ be "$x = x+1$" where $x \in \mathbb{R}$, then $\exists x\, P(x)$ is false

Reason:- No real number = real number + 1

(3) Let $P(x)$ be "$x^2 > 10$" where the domain is integers greater than 4.
Then $\exists x P(x)$ is true

Reason. $x = 5$ and $5^2 = 25 > 10$.

Precedence of Quantifiers.

$\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$

but not $\forall x (P(x) \vee Q(x))$

The quantifiers $\forall$ and $\exists$ have higher precedence than $\neg, \wedge, \vee, \oplus$.

Logical equivalence Involving Quantifiers:-

$\forall x (P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent.

i.e, $\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Negating Quantified expressions.

$$\left.\begin{aligned} \neg \forall x P(x) &\equiv \exists x \neg P(x) \\ \neg \exists x P(x) &\equiv \forall x \neg Q(x) \end{aligned}\right\} \text{De Morgan's laws}$$

Example (1): Negate the statement every student in your class has taken a course in calculus.

Solution: Let $P(x)$ be the statement "x has taken a course in calculus"
Domain is students in your class

Now $\neg \forall x P(x)$ means it is not the case that every student in your class has taken a course in calculus.

It means that there is atleast one student who has not taken a course in Calculus

$\Rightarrow \exists x \neg P(x)$

$\therefore \neg \forall x P(x) \equiv \exists x \neg P(x)$

Similarly $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Example ① $\neg \forall x (x^2 > x)$

Ans $\exists x \neg (x^2 > x)$

$\Rightarrow \exists x (x^2 \not> x)$

$\Rightarrow \exists x (x^2 \leq x)$

② $\neg \exists x (x^2 = 2)$

Ans. $\forall x \neg (x^2 = 2)$

$\Rightarrow \forall x (x^2 \neq 2)$

Example. Show that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (P(x) \wedge \neg Q(x))$ are logically equivalent.

Solution. $\neg \forall x (P(x) \rightarrow Q(x)) \equiv \exists x \neg (P(x) \rightarrow Q(x))$

$\equiv \exists x$ (It is not the case that if $P(x)$ then $Q(x)$)

$\equiv \exists x (P(x) \wedge \neg Q(x))$

Example. Suppose domain has two elements say x_1 and x_2

$\neg \forall x P(x) \equiv \neg (P(x_1) \wedge P(x_2))$

$\equiv \neg P(x_1) \vee \neg P(x_2)$

$\neg \exists x P(x) \equiv \neg (P(x_1) \vee P(x_2))$

$\equiv \neg P(x_1) \wedge \neg P(x_2)$

Translating sentences in English into logical expression
Expressing Statements using Predicates and Quantifiers

Example (1) Express the statement
"Every student in this class has studied calculus"

Ans. $\forall x\, C(x)$ where $C(x)$ is the statement "x has studied calculus" and the domain for x is students in this class.

Example (2) "Some students in this class has visited Mexico"

Sol:- Let $M(x)$ be the statement that "x has visited Mexico" where the domain x is students in this class.
Then given statement is $\exists x\, M(x)$.

Example (3) Every student in this class has visited either Canada or Mexico

Sol Let $C(x)$ be the statement that "x has visited Canada"
Let $M(x)$ be the statement that "x has visited Mexico" where the domain x students in this class
Then given statement is $\forall x\, (C(x) \vee M(x))$

Example (4): Consider the statements

"All lions are fierce"

"Some lions do not drink coffee"

"Some fierce creatures do not drink coffee"

Sol: Let $P(x)$ be the statement

"x is a lion"

Let $Q(x)$ be the statement

"x is fierce"

Let $R(x)$ be the statement

"x drinks coffee"

Where the domain x is all creatures.

Now

All lions are fierce $\Rightarrow \forall x\,(P(x) \to Q(x))$

Some lions do not drink coffee $\Rightarrow \exists x\,(P(x) \wedge \neg R(x))$

Some fierce creatures do not drink coffee $\Rightarrow \exists x\,(Q(x) \wedge \neg R(x))$

*** NOTE**

Some lions do not drink coffee. We cannot write this $\exists x(P(x) \to \neg R(x))$

Reason: $P(x) \to \neg R(x)$ is true when $P(x)$ is false $\Rightarrow x$ is not a lion

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

When we write $\exists x(P(x) \to \neg R(x))$ then this is true for some creature that is not a lion, even if every lion drinks coffee.

Nested Quantifiers

Let $P(x,y)$ be a statement.

$\forall x \forall y\, P(x,y)$ means for all x, for all y such that $P(x,y)$

$\forall y \forall x\, P(x,y)$ means for all y, for all x such that $P(x,y)$

$\forall x \exists y\, P(x,y)$ means for all x, there is a y such that $P(x,y)$

* $\exists x \forall y\, P(x,y)$ means there exists an x such that $P(x,y)$ for all y

$\left.\begin{array}{l}\exists x \exists y\, P(x,y) \\ \exists y \exists x\, P(x,y)\end{array}\right\}$ means there is a pair x, y such that $P(x,y)$.

Examples $\boxed{\text{Translate into English from logic}}$

1) $\forall x \forall y\, (x+y = y+x)$ where the domain for x and y are all real numbers.

Ans. For ~~every~~ all real number x and for all real numbers y $x+y = y+x$ (Commutative law for addition)

2) $\forall x \exists y\, (x+y=0)$

Ans For all real numbers x, there is a real number y such that $x+y=0$ (Additive Inverse).

3) $\forall x \forall y \forall z\, (x+(y+z) = (x+y)+z)$

Ans. For all real numbers x, y, z, $x+(y+z) = (x+y)+z$ (Associative law for addition)

4) Translate into English

$$\forall x \forall y \left((x>0) \wedge (y<0) \rightarrow (xy<0) \right)$$

where domain for x and y is $\mathbb{R}$

Ans. For every real number x and for every real number y, if $x>0$ and $y<0$ then $xy<0$

5) Let $P(x,y)$ be the statement "$x+y=y+x$"

What is the truth value of the quantifications (i) $\forall x \forall y\, P(x,y)$ (ii) $\forall y \forall x\, P(x,y)$.

where domain for x and y is $\mathbb{R}$

Ans

$\forall x \forall y\, P(x,y)$. means for all real numbers x and for all real numbers y, $x+y=y+x$.

$\Rightarrow \forall x \forall y\, P(x,y)$ is true ($\because$ For real numbers $x+y=y+x$)

Similarly $\forall y \forall x\, P(x,y)$ is true.

6) Let $P(x,y)$ be the statement "$x+y=0$" what are the truth values of the quantifications (i) $\exists y \forall x\, P(x,y)$ (ii) $\forall x \exists y\, P(x,y)$ when domain for x and y is $\mathbb{R}$.

Sol (i) $\exists y \forall x\, P(x,y)$ means there is atleast one y such that $x+y=0$ for every x

$\Rightarrow \exists y \forall x\, P(x,y)$ is False ($\because$ there is only one $y=-x$ for every x such that $x+y=0$ when $x,y \in \mathbb{R}$)

(ii) $\forall x \exists y\, P(x,y)$ means for all x there is a y such that $x+y=0$

$\Rightarrow \forall x \exists y\, P(x,y)$ is True ($\because$ For every x, there is $y=-x$ such that $x+y=0$

7) Let $P(x, y, z)$ be the statement "$x+y=z$"
what are the truth values of
(i) $\forall x \forall y \exists z P(x, y, z)$
(ii) $\exists z \forall x \forall y P(x, y, z)$
where domain of x, y, z is $\mathbb{R}$

Sol

(i) $\forall x \forall y \exists z P(x, y, z)$ means for every x and for every y there is atleast one z such that $x+y=z$

$\Rightarrow \forall x \forall y \exists z P(x, y, z)$ is True

(ii) $\exists z \forall x \forall y P(x, y, z)$ means there exists atleast one z such that $x+y=z$ for every x and for every y

$\Rightarrow \exists z \forall x \forall y P(x, y, z)$ is False
(Reason: there is no value of z which satisfies $x+y=z$ for all values of x and y)

Translate into logic from English

Example ① "The sum of two positive integers is always positive"

Translate this to logic

Solution $\forall x \forall y \left((x>0) \wedge (y>0) \rightarrow (x+y>0) \right)$

where the domain for x and y is Z^+

② "Every real number except zero has a multiplicative inverse"

Solution

$\forall x \left((x \neq 0) \rightarrow \exists y (xy=1) \right)$

(Definition of multiplicative inverse)

A multiplicative inverse of a real number x is a real number y such that $xy=1$

Negating Nested Quantifiers

Negate the statement $\forall x \exists y (xy=1)$

Sol $\neg \forall x \exists y (xy=1) \equiv \exists x \neg \exists y (xy=1)$

$\equiv \exists x \forall y \neg (xy=1)$

$\equiv \exists x \forall y (xy \neq 1)$

Problems (Important conclusions).

1) The universal quantification $\forall x\, P(x)$ is true when $P(x)$ is true for every x.

2) The universal quantification $\forall x\, P(x)$ is false when there is an x for which $P(x)$ is false.

3) The existential quantifier $\exists x\, P(x)$ is true when there is an x for which $P(x)$ is true.

4) The existential quantifier $\exists x\, P(x)$ is false when $P(x)$ is false for every x.

5) $$\neg \exists x \big(P(x) \wedge Q(x)\big) \equiv \forall x\, \neg\big(P(x) \wedge Q(x)\big)$$
$$\equiv \forall x\, \big(P(x) \rightarrow \neg Q(x)\big)$$

6) NOTE
 1) $p \rightarrow q \equiv \neg p \vee q$
 2) $p \rightarrow q \equiv \neg q \rightarrow \neg p$
 3) $p \vee q \equiv \neg p \rightarrow q$
 4) $p \wedge q \equiv \neg (p \rightarrow \neg q)$
 5) $\neg (p \rightarrow q) \equiv p \wedge \neg q$

7) Consider the statements. Let $P(x)$ be the statement that x is even, $Q(x)$ be the statement that x is a prime number and $R(x)$ be the statement that 5 divides x. Then translate 1) $\exists x\, (P(x) \wedge Q(x))$ 2) $\forall x\big((P(x) \wedge Q(x)) \rightarrow R(x)\big)$ into english statements. Where the domain is integers.

Ans 1) $\exists x\,(P(x) \wedge Q(x))$ — There exist atleast one integer x such that x is even and x is prime.

2) $\forall x\big((P(x) \wedge Q(x)) \rightarrow R(x)\big)$ — For all integers x, if x is even and prime then 5 divides x.

Rules of Inference for Propositional Logic

Rule of Inference	Tautology
1) Premises ← p, $p \to q$; Conclusion ← $\therefore q$	$(p \wedge (p \to q)) \to q$
2) $\neg q$, $p \to q$, $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$.
3) $p \to q$, $q \to r$, $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
4) $p \vee q$, $\neg p$, $\therefore q$	$((p \vee q) \wedge \neg p) \to q$
5) p, $\therefore p \vee q$	$p \to (p \vee q)$
6) $p \wedge q$, $\therefore p$	$(p \wedge q) \to p$
7) p, q, $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$
8) $p \vee q$, $\neg p \vee r$, $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$

Introduction to Proofs:-

Definitions

1) Theorem: A theorem is a statement that can be shown to be true

2) Proof: A proof is a valid argument that establishes the truth of a theorem.

3) Lemma:- A less important theorem that is helpful in the proof of other results is called a lemma

4) Corollary:- A corollary is a theorem that can be established directly from a theorem that has been proved

5) Conjecture:- A conjecture is a statement that is being proposed to be a true statement, usually on the basis of some partial evidence or the intuition of an expert.

When a proof of a conjecture is found, the conjecture becomes a theorem.

Methods of Proving Theorems

1) Direct Proofs
2) Indirect Proofs - Proof by Contraposition.
3) Proofs by Contradiction.

Direct Proof of Conditional Statement $p \to q$

In a direct proof, we assume that p is true and we use axioms and other related properties to show that q must also be true.

Example:- Prove that "if n is odd integer, then n^2 is odd.

Solution. Assume that n is odd

$\Rightarrow n = 2K+1$

To show that n^2 is odd.

Now $n^2 = (2K+1)^2$

$= 4K^2 + 4K + 1$

$= 2(2K^2 + 2K) + 1$

$=$ Even integer $+ 1$

$=$ odd integer.

NOTE:
Even integer n means $n = 2K$
odd integer n means $n = 2K+1$
for some integer K

NOTE:- Some times direct proofs will not work for all problems.
For example prove that "if n^2 is odd, then n is odd"

Sol Assume that n^2 is odd

$\Rightarrow n^2 = 2K+1$

$\Rightarrow n = \pm\sqrt{2K+1}$

So we cannot conclude what type of n is this.

In such cases we use indirect proof called Proof by contraposition

* The conditional statement $p \to q$ is equivalent to its contrapositive $\neg q \to \neg p$.

Example Prove that "If n^2 is odd, then n is odd" for any integer n.

Sol : Let p be the statement n^2 is odd
q be the statement n is odd.

Then "If n^2 is odd, then n is odd"

$\Rightarrow p \longrightarrow q$

The contraposition is $\neg q \rightarrow \neg p$.

That means we have to prove that "if n is even then n^2 is even"

Assume that n is even

$\Rightarrow n = 2k$

Now $n^2 = (2k)^2$
$= 4k^2$
$= 2(2k^2)$
$=$ Even.

We have proved "if n is even then n^2 is even"
The Contraposition is "If n^2 is odd then n is odd"

Example:- Prove that Sum of two rational numbers is Rational.

Sol Rational number means a number in the form of $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.

$\boxed{\text{Direct Proof Attempt}}$

Let $r = \frac{p}{q}$ and $s = \frac{l}{m}$ be two Rational numbers where $q \neq 0$, $m \neq 0$

then $r + s = \frac{p}{q} + \frac{l}{m}$

$$r+s = \frac{mp+qr}{qm}.$$

$\Rightarrow r+s =$ Rational number. (Since $q \neq 0, m \neq 0$ $\Rightarrow qm \neq 0$)

$\Rightarrow$ Sum of two rational numbers is rational number.

Proof by CONTRADICTION:-

Suppose we want to prove that p is true.

If we can find a contradiction q such that $\neg p \rightarrow q$ is true, then p will be true.

(The reason is: $\neg p \rightarrow q$ is true means $\neg p$ must be false as q is false. As $\neg p$ is false $\Rightarrow p$ is true)

p	q	$p \rightarrow q$
T	T	F
T	F	F
F	T	T
F	F	T

Example:- Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition "$\sqrt{2}$ is irrational".

Proof by contradiction

suppose that $\neg p$ is true.

$\Rightarrow \sqrt{2}$ is rational

$\Rightarrow \sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$

$\Rightarrow$ Squaring on both sides

$$2 = \frac{a^2}{b^2}$$

$\Rightarrow 2b^2 = a^2$

$\Rightarrow a^2$ is even (Since $2b^2$ is even).

As a^2 is even

$\Rightarrow$ a is even

$\Rightarrow$ $a = 2K$

Now. $2b^2 = a^2$

$\Rightarrow 2b^2 = (2k)^2$

$\Rightarrow 2b^2 = 4K^2$

$\Rightarrow b^2 = 2K^2$

$\Rightarrow$ b^2 is even (Since $2K^2$ is always even)

$\Rightarrow$ b is even.

As both a and b are even, 2 divides both a and b.

But $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}, b \neq 0$ and a and b have no common factors

~~$\Rightarrow \sqrt{2} = \frac{\text{Even number}}{\text{Even number}}$~~

$\Rightarrow$ 2 does not divide both a and b.

This is a contradiction to the fact that $\sqrt{2}$ is rational.

$\Rightarrow \neg p$ is false.

$\Rightarrow p$ is true

$\Rightarrow \sqrt{2}$ is irrational