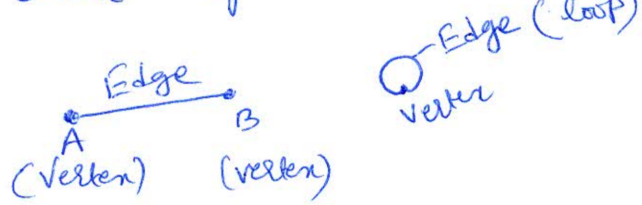
GRAPHS

CHAPTER-10

WEEK-13

GRAPH :- (UNDIRECTED) : A graph G consists of a set of nonempty vertices or nodes (V) and a set of edges E. i.e, $G = (V, E)$

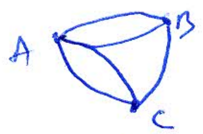
* Each edge has either one or two vertices associated with it, called endpoints



Loop :- An edge that connect a vertex to itself

Simple graph :- A graph in which each edge connects two different vertices and no two edges connect the same pair of vertices is called a simple graph → UNDIRECTED

Simple.



Not Simple.
(Multiple edges)

Not Simple.

Multiple edges :- Edges connecting the same pair of vertices

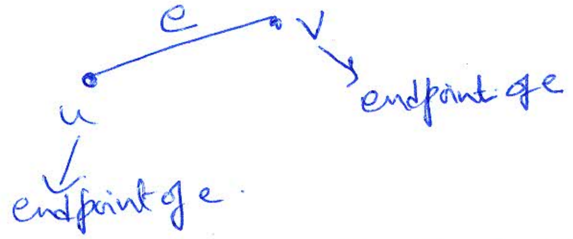
Multiple edges.

Pseudo-graph :- Graphs having loops and multiple edges are called Pseudograph..

TERMINOLOGY (undirected graphs)

ADJACENT VERTICES (OR) NEIGHBORS :-

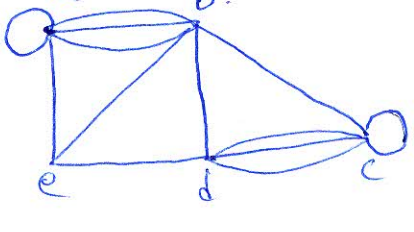
Two vertices u and v of an undirected graph G are called adjacent (or) neighbors in G, if u and v are endpoints of an edge e of G.



Here edge e is called incident with u and v and e connects u and v.

Neighborhood of vertex :- The set of all neighbors of a vertex v of graph G, is called the neighborhood of v and is denoted by N(v).

Example



$N(a) = \{a, b, e\}$
$N(b) = \{a, b, c, d, e\}$
$N(c) = (b, c, d)$
$N(d) = (b, c, e)$
$N(e) = \{a, b, d\}$

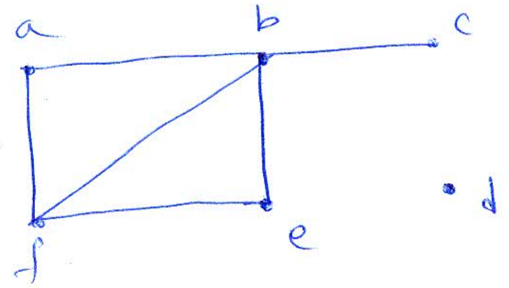
DEGREE OF A VERTEX :-

The number of edges incident with the vertex is called the degree of a vertex. It is denoted by deg(v).

* Loop at a vertex contributes twice to the degree of that vertex.
* A vertex of degree zero is called isolated vertex.
* A vertex is PENDANT if and only if it has degree one.

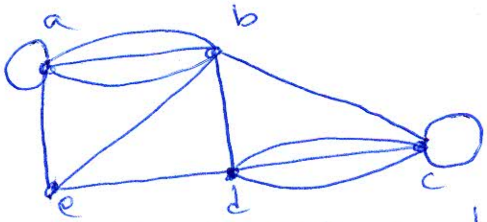
Examples

(1)



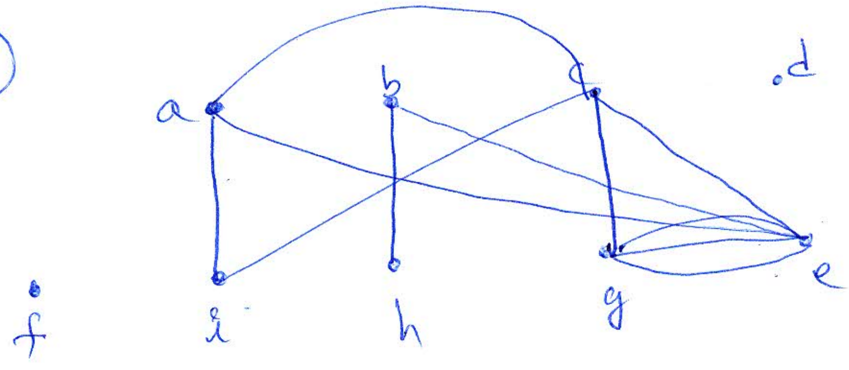
deg(a) = 2, deg(b) = 4, deg(c) = 1, deg(d) = 0.
deg(e) = 2, deg(f) = 3.
vertex d is isolated
vertex c is pendant.

(2)



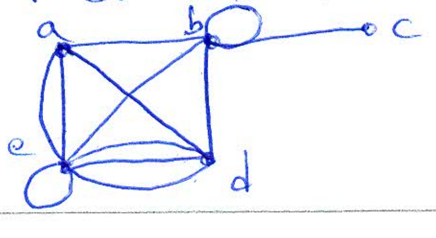
deg(a) = 6, deg(b) = 6, deg(c) = 6, deg(d) = 5, deg(e) = 3
No vertex is isolated and No vertex is Pendant

(3)



deg(a) = 3, deg(b) = 2, deg(c) = 4, deg(d) = 0, deg(e) = 6
deg(f) = 0, deg(h) = 1, deg(i) = 2
Isolated vertices are d, f.
Pendant is vertex h.

(4)

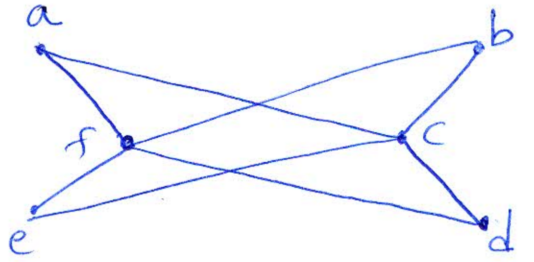


deg(a) = 4   deg(d) = 5
deg(b) = 6   deg(e) = 8
deg(c) = 1   vertex c is Pendant.

THE HANDSHAKING THEOREM:- Let $G = (V, E)$ be an undirected graph, then sum of degrees of all vertices of G is (even) two times the edges.

i.e., $\sum_{v \in V} \deg(v) = 2m$, where m is the number of edges.

Example 1) verify handshaking theorem for the graph.



Sol

Number of edges $m = 8$

$\deg(a) = 2$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 2$

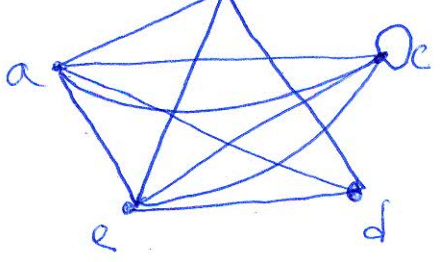
$\deg(e) = 2$, $\deg(f) = 4$

$\sum_{v \in V} \deg(v) = 2 + 2 + 4 + 2 + 2 + 4 = 16.$

$2m = 2(8) = 16$

$\Rightarrow \sum_{v \in V} \deg(v) = 2m$

2) Verify Handshaking theorem for the graph.



Solution

Number of edges $m = 12$

$\deg(a) = 5$

$\deg(b) = 5$

$\deg(c) = 6$

$\deg(d) = 3$

$\deg(e) = 5$

$\sum_{v \in V} \deg(v) = 24$

and $2m = 2(12) = 24$

Theorem:- An undirected graph has an EVEN numbers of vertices of odd degree.

$$2m = \sum_{v \in V} \deg(v)$$

$$\Rightarrow 2m = \underbrace{\sum_{v \in V_1} \deg(v)}_{\text{Sum of degrees of vertices where degree of each vertex is odd}} + \underbrace{\sum_{v \in V_2} \deg(v)}_{\text{Sum of degrees of vertices where degree of each vertex is even}}$$

$$\Rightarrow 2m = \underbrace{\text{Even number/odd number}}_{\text{This term becomes an even number when we add even number of odd degrees}} + \text{Even number}$$

DIRECTED GRAPH (DIGRAPH) :- A directed graph or digraph (V, E) consists of nonempty set of vertices and set of directed edges (or) nodes.

* Each directed edge is associated with an ordered pair of vertices.

* Directed edge associated with the ordered pair (u, v) is said to start at u and end at v. u is called initial vertex and v is terminal vertex of (u, v).
u is said to be adjacent to v
v is said to be adjacent from v

* For a Loop, the initial vertex and terminal vertex are same.

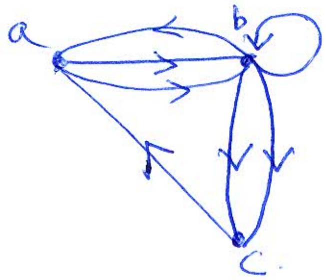
NOTE:- In undirected graphs, each undirected edge is associated to an unordered pair of vertices, whereas in directed graphs, each directed edge is associated to an ordered pair of vertices.

IN-DEGREE OF A VERTEX:- In a directed graph, the in-degree of a vertex is the number of edges with v as their terminal vertex. It is denoted by $\deg^-(v)$.

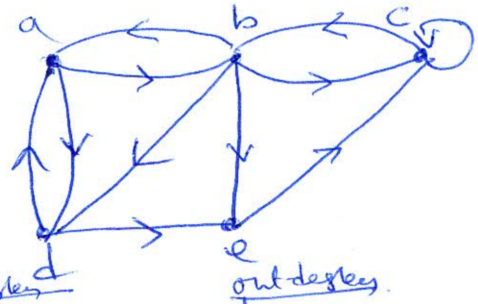
OUT-DEGREE OF A VERTEX:- The out-degree of a vertex is the number of edges with v as their initial vertex. It is denoted by $\deg^+(v)$.

* Loop at a vertex contributes 1 to in-degree and 1 to out-degree of this vertex.

Examples:-



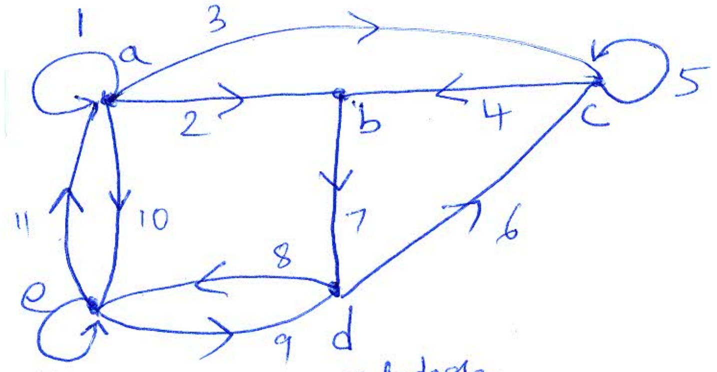
Indegrees	out degrees
$\deg^-(a) = 2$	$\deg^+(a) = 2$
$\deg^-(b) = 3$	$\deg^+(b) = 4$
$\deg^-(c) = 2$	$\deg^+(c) = 1$



In-degrees	out degrees
$\deg^-(a) = 2$	$\deg^+(a) = 2$
$\deg^-(b) = 2$	$\deg^+(b) = 4$
$\deg^-(c) = 3$	$\deg^+(c) = 2$
$\deg^-(d) = 2$	$\deg^+(d) = 2$
$\deg^-(e) = 2$	$\deg^+(e) = 1$

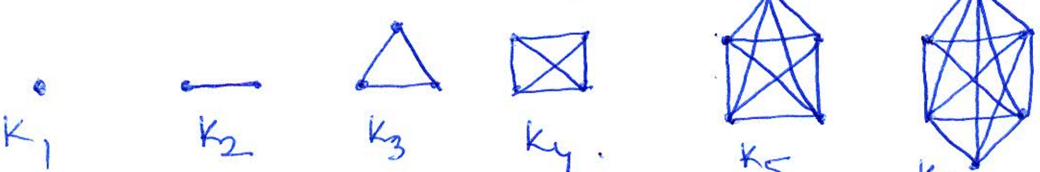
THEOREM:- If $G = (V, E)$ is a directed graph, then $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$.

Eg



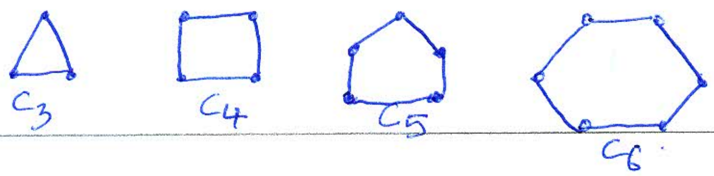
In-degrees	out-degrees	Number of edges, E
$\deg^-(a) = 2$	$\deg^+(a) = 4$	
$\deg^-(b) = 2$	$\deg^+(b) = 1$	
$\deg^-(c) = 3$	$\deg^+(c) = 2$	
$\deg^-(d) = 2$	$\deg^+(d) = 2$	
$\deg^-(e) = 3$	$\deg^+(e) = 3$	
12	12	12

COMPLETE GRAPH (K_n):- A simple graph that contains exactly one edge between every pair of 'n' distinct vertices. It is denoted by K_n.



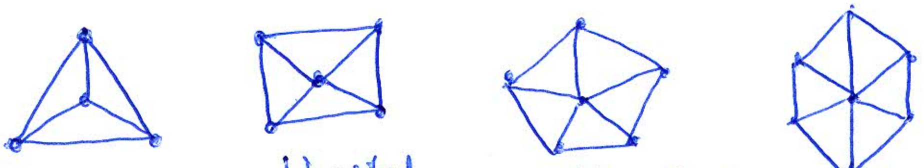
K_1 K_2 K_3 K_4. K_5 K_6

CYCLE:- A cycle C_n ($n \geq 3$) consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{n-1}, v_n\}$ & $\{v_n, v_1\}$



C_3 C_4 C_5 C_6.

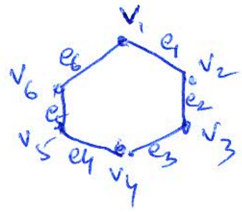
WHEEL:- A wheel W_n is obtained from a cycle C_n $(n \geq 3)$ by adding an additional vertex, which connects each of the n vertices of the cycle.

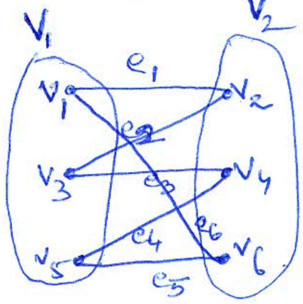


Wheel - W_3	W_4 - wheel	W_5 - wheel	wheel - W_6
Vertices - 4	vertices - 5	vertices - 6	vertices - 7
Edges - 6	Edges - 8	Edges - 10	Edges - 12

BIPARTITE GRAPH:- A simple graph G is called a bipartite graph if the vertex set V can be partitioned into TWO DISJOINT sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

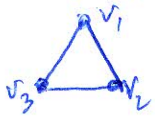
Example 1) Consider the cycle C_6.





The cycle C_6 is bipartite.

2) The complete graph K_3 is not bipartite.



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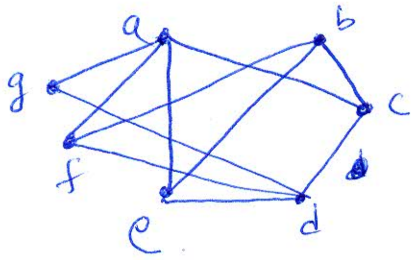
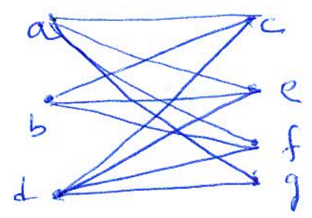




Bipartite

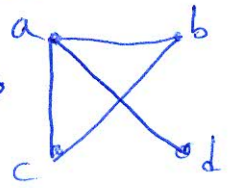
REPRESENTING GRAPHS.

ADJACENCY MATRIX: If $G = (V, E)$ is a simple graph with n vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix A_G of the graph G with respect to listing of the vertices is a $n \times n$ zero-one matrix with 1 as its $(i,j)^{th}$ entry when the vertices v_i and v_j are adjacent otherwise zero as its $(i,j)^{th}$ entry when v_i and v_j are not adjacent.

i.e, $a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0, & \text{otherwise.} \end{cases}$ → unordered pair.

Examples

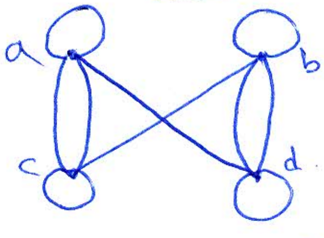
Graph →



Adjacency Matrix

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Pseudo graph.



Adjacency Matrix

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$

NOTE

1) A Pseudograph can also be represented by an adjacency matrix which is not a zero-one matrix.

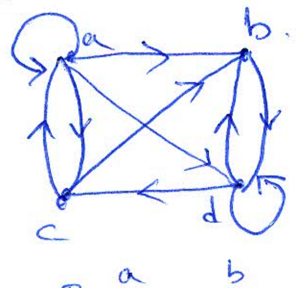
2) For undirected graphs, the adjacency matrix is always an symmetric matrix.

3) Adjacency matrix can also be used to represented directed graphs.

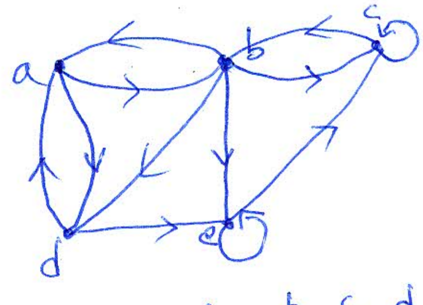
The matrix for a directed graph $G = (V, E)$ has a 1 in its $(i, j)^{th}$ position if there is an edge from v_i to v_j, otherwise 0 in $(i, j)^{th}$ position if there is no edge from v_i to v_j.

$$a_{ij} = \begin{cases} 1 & \text{if } \underbrace{(v_i, v_j)}_{\text{ordered pair}} \text{ is an edge of } G \\ 0, & \text{otherwise.} \end{cases}$$

Examples.



$$A_G = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0 & 0 & 0 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 1 & 1 & 1 \end{array}$$



$$A_G = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 0 & 1 & 0 \\ b & 1 & 0 & 1 & 1 & 1 \\ c & 0 & 1 & 1 & 0 & 0 \\ d & 1 & 0 & 0 & 0 & 1 \\ e & 0 & 0 & 1 & 0 & 1 \end{array}$$

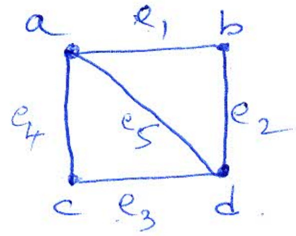
NOTE 1) The adjacency matrix for a directed graph need not be symmetric.

2) The adjacency matrix for a directed graph need not be a zero-one matrix, when there are multiple edges in the same direction.

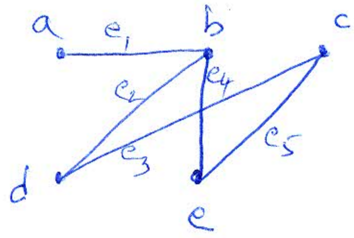
INCIDENCE MATRICES:- Let $G = (V, E)$ be an undirected graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. The incidence matrix with respect to this ordering of V and E is an $n \times m$ matrix.

$$M_G = [m_{ij}]_{m \times n} \quad \text{where} \quad m_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with vertex } v_i \\ 0, & \text{otherwise.} \end{cases}$$

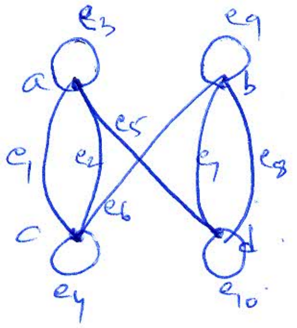
Examples:-

1) 

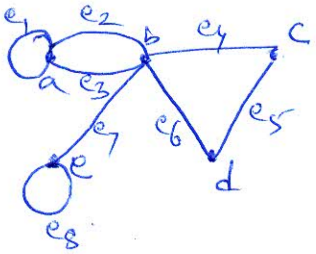
$$M_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

2) 

$$M_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

3) 

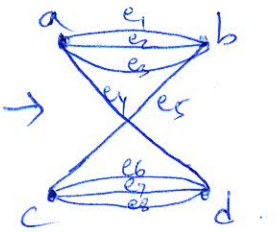
$$M_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

4) 

$$M_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Examples:-

1) Graph →



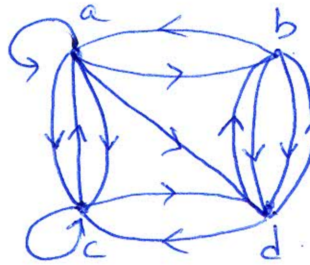
→ Adjacency Matrix

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

Incidence matrix

$$M_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

2)

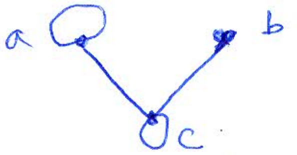


Adjacency Matrix

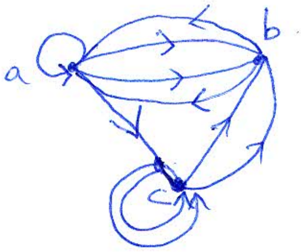
$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

3) Draw the graph represented by the given adjacency matrix.

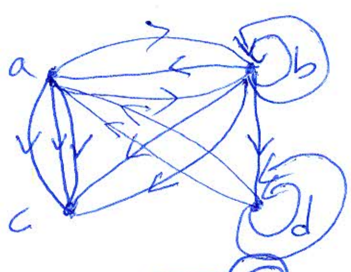
$$A_G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow$$



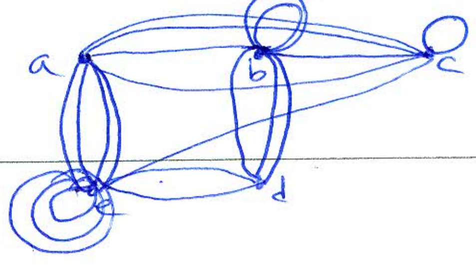
$$A_G = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} \rightarrow$$



$$A_G = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$



$$A_G = \begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

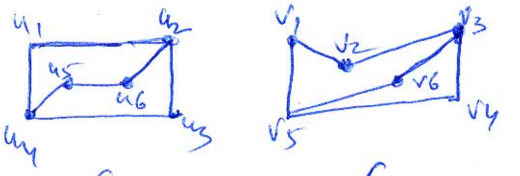


ISOMORPHISM OF GRAPHS:- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a one-to-one and onto function from V_1 to V_2 such that ~~f~~, ~~the~~ the vertices a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2.

* Isomorphic simple graphs must have the same number of vertices
* Isomorphic simple graphs also must have the same number of edges.
* The degrees of the vertices in isomorphic simple graphs must be same. That is, a vertex 'v' of degree 'd' in G_1 must correspond to a vertex f(v) of degree d in G_2

Examples

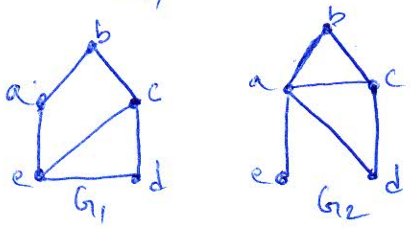
1)



G_1 G_2

— ISOMORPHIC

2)



— Not isomorphic.

G_1 – edges – 6
G_2 – edges – 6

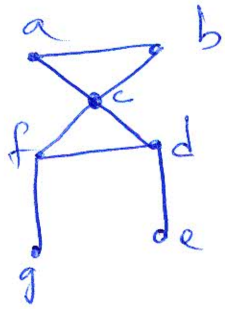
G_2 has one vertex e of degree 1. But G_1 does not have any vertex of degree 1

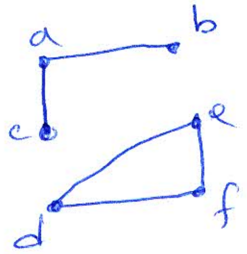
PATH:- A PATH is a sequence of edges that begins at a vertex of a graph and travels from one vertex to ~~other~~ vertex along edges of the graph.

CIRCUIT:- The Path is a circuit if it begins and ends at the same vertex and has length greater than zero

CONNECTED:- An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.

Example

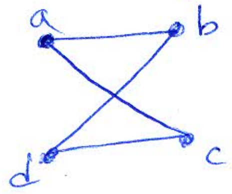
1) The graph  is connected

2) The graph  is not connected.

COUNTING PATHS BETWEEN VERTICES :

Let G be a graph with adjacency matrix A_G with respect to ordering of vertices $v_1, v_2, \ldots v_n$. The numbers of paths of length r from v_i to v_j is the $(i, j)^{th}$ entry of A_G^r

Eg How many paths of length 4 are there from a to d in the graph G.



Sol The Adjacency matrix $A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$

$$(A_G)^4 = \begin{bmatrix} 8 & 0 & 0 & \underline{\underline{8}} \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Number of paths of length 4 from a to d is $(1, 4)^{th}$ entry of $(A_G)^4 = 8$